Runge-Kutta Order 2

Taylor Series of $f(x, y)$

Recall in class we expanded $y(x+h)$ in a Taylor series about x, and to help you remember this the first thing I want you to do is expand the first three terms of a function $g(x+h)$ in a Taylor series about x.

$g(x+h)=$____(1)

With this as a blue print now see if you can't expand the first three terms of a function $m(y+k)$ in a Taylor series about y.

$m(y+k)=$____(2)

In mathematics we often use subscripts to represent partial derivatives, for example , $\frac{\partial^3 f(x,y)}{\partial x^3} = f_{xxx}$. See if you can simplify the following three; be careful on the last one.

A) $\frac{\partial^2 f(x,y)}{\partial y^2} =$ ________ B) $\frac{\partial^3 f(x,y)}{\partial x^2 \partial y} =$ ________ C) $\frac{\partial f(x,y)}{\partial z} =$ ________

Now let's see if you know how to take partial derivatives. If $f(x,y) = x^2 y + \cos(y^2)$ then find

A) $f_{xx} =$ _____ B) $f_{xy} =$ _____

Using equation (1) as a guide, what do you think that the first three terms of the Taylor series of the function $g(x+h) = 5n''(x+h)$ about x would look like? Write these in terms of the derivatives of n and not g.

$5n''(x+h) =$____(3)

Now we are ready to learn what a Taylor series of a function of two variables $f(x+h, y+k)$ expanded about the point (x, y) will look like. The trick to developing this is to hold each variable fixed while you expand a Taylor series about the other. Turn the page and I will try to explain.

First we will hold the $y+k$ fixed and pretend, cover up the $y+k$ with your finger if you must, that f is a function of $x+h$ only. So now find the first four terms of Taylor series of the first coordinate of $f(x+h, y+k)$ about x. Let the second coordinate $y+k$ just hang around. Write down the $y+k$ but ignore it. Use the subscript notation for the derivatives of x since these are really partial derivatives. Use equation (1) as your guide. I will start you off.

$$f(x+h, y+k) = f(x, y+k) + f_x(x, y+k)h + \underline{\hspace{6cm}} \quad (4)$$

In the end we are only going to want just the very first few terms of the two dimensional Taylor series of $f(x+h, y+k)$, but how to get those early terms is the question. Well, now comes the interesting part we need to expand each of the four terms (*functions*) listed on the right hand side of equation (4) in a Taylor series about y where now we keep the x coordinates fixed. *SCARY*! *I know, but I will help you.* Let's start with the easiest of the four: the first term. So here we go (*Matt if you are really reading this I will be shocked!)* Get the first four terms of this T.S. about y. I will start

$$f(x, y+k) = f(x, y) + f_y(x, y)k + \underline{\hspace{6cm}} \quad (5)$$

Good, or at least I think good. Feeling braver? then go for the second term. You may want to look at equation (3) to get a feel for the idea. Get the first three terms of this T.S. about y.

$$hf_x(x, y+k) = f_x(x, y)h + \underline{\hspace{8cm}} \quad (6)$$

Go ahead and get the first two terms of the T.S. of the third and fourth terms of the right hand side of (4) about y.

$$\frac{h^2}{2} f_{xx}(x, y+k) = f_{xx}(x, y)\frac{h^2}{2} + \underline{\hspace{5cm}} \quad (7)$$

$$\frac{h^3}{6} f_{xxx}(x, y+k) = f_{xxx}(x, y)\frac{h^3}{6} + \underline{\hspace{5cm}} \quad (8)$$

If we would add up all of the terms in (5)-(8), this would represent some of the terms in the two dimensional Taylor series of $f(x+h, y+k)$, but let's do this in an organized way. Go back through all the terms in (5)-(8) and find all the 0^{th} order terms in h and k. This means all the terms that aren't multiplied by any multiple of h or k. List this/these now.

0^{th} order____ (9)

Now go back through all the terms in (5)-(8) and find all the 1^{st} order terms in h and k. This means all the terms that are multiplied by either h alone or k alone. List this/these now.

1^{st} order____ (10)

Now go back through all the terms in (5)-(8) and find all the 2^{nd} order terms in h and k. This means all the terms that are multiplied by either h^2 alone, k^2 alone, or hk alone. List this/these now.

2^{nd} order____ (11)

Finally go back through all the terms in (5)-(8) and find all the 3^{rd} order terms in h and k. See if you can figure out what forms these can take. List this/these now.

3^{rd} order____ (12)

Starting with the terms in (9) and moving sequentially to the terms in (12), write down what represents the lowest order terms of the two-dimensional Taylor series of $f(x+h, y+k)$ expanded about the point (x, y):

$f(x+h, y+k) =$ ____________________________________

____ (13)

____

+ *O(* 4^{th} order terms in h and k*)*

With (14) developed we can start looking at solving 1^{st} order *IVP's* again, we want an approximate solution to the general form of the *IVP*

$$y'(x) = f(x, y(x)) \quad \text{subject to} \quad y(x_0) = y_0 \tag{14}$$

We have already developed accurate numerical schemes for (14), called higher order Taylor series methods, but these require that you calculate the derivatives of f, the right hand side of (14). These derivatives can be quite cumbersome as I am certain you have noticed. If not, then I have been too easy on you lately. Runge-Kutta methods, while looking worse at the beginning, are very nice in that they are the same order accuracy as T.S. methods but they do not require you to take the derivatives of *f*. The goal of the remainder of this worksheet is to introduce a Runge-Kutta order 2 method and to show you why it is in fact an order two method.

"I bet you just blew right through what I just wrote just so that you can get out of here a little earlier?" Go back to the TOP of the page and READ IT AGAIN! You make me so furious when you think la la la let me finish this worthless little worksheet, I'm hungry, what am I going to do for lunch, I can't believe I have another #$%^%$^%$ PA to finish #$%#%^$. READ IT AGAIN!

The way that we are going to accomplish our goal is by showing the series expansion of the Runge-Kutta order 2 method can be manipulated into the same form as 2nd order T.S. method we talked about the other day. What we will do is light the candle on both ends and meet at the middle. Let's start with the one you already know a little about. T.S. method order 2. List this method now.

"WHAT do you MEAN you don't remember the method?" It is moments like this that make me feel that my life has such meaning.

OK, lets take it from the top. Expand $y(x+h)$ using Taylor's theorem with remainder about x. Make your third derivative term the remainder.

$y(x+h) =$ ____ (15)

Now, by (14) we can substitute ___________ for y'. The y'' term is going to require some thought. Before we always had a specific *f*, but not now. In Calculus III you have learned or will learn that

$$\frac{d}{dx}\left[g\left(u(x),w(x)\right)\right] = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial w}\frac{dw}{dx} = g_u u' + g_w w' . \qquad (16)$$

Result (16) is an extension of the chain-rule to two dimensions. Let's use (16) to help us with our y'' term. Then, using (16)

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(f(x, y(x))\right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (17)$$

However, $y' = f$, so make this substitution for the y' term that most of you have in (17), which gives

$$y'' = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (18)$$

Now, in (15), substitute f for y' and (18) for y'', yielding

$$y(x+h) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + O(h^3), \qquad (19)$$

where we have simply listed the remainder term by its order. Equation (19) represents the candle burned from the top (T.S.). Next, we are going to match (19) by burning the candle from the bottom (R-K).

Take a look at this HORRIFIC expression!!!

$$\boxed{y(x) + c_1 h\, f\left(x, y(x)\right) + c_2 h\, f\left(x + c_3 h, y(x) + c_4 h\, f\left(x, y(x)\right)\right)} \qquad (20)$$

AHHHHHHHHHHHHHHHHHHH!!!!!!!!!!!!!! This is the form of a Runge-Kutta order 2 method. The $c_1, c_2, c_3,$ and c_4 are 4 constants to be determined. I know that (20) is a little overwhelming, so we will get a little more comfortable with it first. Suppose that

$$f(x, y(x)) = y - x^2 + 1, \qquad y(1) = 3,$$

$c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$, $h = .5$, and $x = 1$. What number is (20) in this case?

Answer________

Now since you are a little more comfortable with (20), (pretend you are a little more comfortable, humor me to make me think the hours I spent creating this has been ever so valuable) compare (20) to the R.H.S. of (19). The first two terms are very similar with the exception of the c_1. The last terms look, well, nothing alike, but the are actually very similar. YOU DOUBT ME?

ROLL up your sleeves and hold on to your seats!! Expand the third term of (20) in a **two dimensional Taylor series**. Those with heart conditions or one of the other reasons you can't ride roller coasters, (nobody in this room of course), can hop off now. First, we will clean up (20) a little by hiding the independent variable. By the way, what is the independent variable?? ____ Substitute y for $y(x)$ and f for $f(x, y(x))$ in (20). Then notationally, (20) simplifies to

____ (21)

So now the third term in (21) should be looking like

$$c_2 h\, f\left(x + c_3 h, y + c_4 h\, f\right), \tag{22}$$

which looks a little more tractable. For sanity, we will only do our two dimensional T.S. out to 1^{st} order terms. Go back to (13) and rewrite (13) using only 1^{st} order and lower terms.

$f(x+h, y+k) =$ ______________________________________ $+O(\textit{order 2})$ (23)

Using (23) as a guide with $h = c_3 h$ and $k = c_4 hf$ the two dimensional Taylor series of $f\left(x + c_3 h, y + c_4 h\, f\right)$ expanded about the point (x, y) to 1^{st} order terms will be,

$f\left(x + c_3 h, y + c_4 h\, f\right) =$

____ $+O(\,?\,)$ (24)

The order of (24) can be expressed a little easier than the order in (23) where we just noted it as $O(\textit{order 2})$. Since the person, ok thing, playing the role of k in equation (24) really has an h in it also, all second order terms in (24) will be

$$O(h^{\_\_?})$$

Thus, using (24), the two dimensional Taylor series of $c_2 h\, f\left(x + c_3 h, y + c_4 h\, f\right)$ expanded about the point (x, y) to 1^{st} order terms follows by just multiplying (24) by $c_2 h$:

$c_2 h f\left(x + c_3 h, y + c_4 h f\right) =$

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + O(h^{—?}) \quad (25)$$

Substitute the R.H.S. of (25) for $c_2 h f\left(x + c_3 h, y + c_4 h f\right)$ in (21), which makes our Runge-Kutta order 2 method look like,

$$y + c_1 hf + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + O(h^{—?}) \quad (26)$$

In (26), set $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}, c_3 = 1, \text{and } c_4 = 1$, which in turn makes our Runge-Kutta order 2 method look like,

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + O(h^{—?}) \quad (27)$$

Compare (27) to the R.H.S. of (19). What do you see?______________

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What does this mean? Recall that (27) represents the 2 dimensional T.S. expansion of (20), and that this produced an identical expansion to the 2nd order T.S. method (19). This means that, to order h^3, you can get the same accuracy in your numerical method by approximating $y(x+h)$, your next step, according to (20), as you can by computing according to (19). Of course, the constants in (20) are taken to be $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}, c_3 = 1, \text{and } c_4 = 1$ if we simply want to recover T.S. order 2 method. The advantage (20) has over (19) is that at NO TIME in (20) do you need to take a derivative of f. You have traded taking a derivative for taking an extra clever function evaluation, which shows up in the guts of the third term. Other methods (Heun's, Modified Euler, Midpoint, …) use different values for the c_i's.

Higher order Runge-Kutta methods are developed similarly. Runge-Kutta order 4 is generally the standard technique for numerical solution of IVPs and systems involving differential equations…..until, of course, Drs. Parker and Sochacki's Picard method takes the World by storm….

Now, solve our good old buddy $y' = y - x^2 + 1 \quad y(0) = .5$ on [0,2] with n=10 steps, but this time take your sequence of steps using the Runge-Kutta order 2 method we just created.