39] Show for any $K>0$, $\ln(x) = O(x^K)$ as $x \to \infty$

Using defn p.19

$$f(x) = O(g(x)) \text{ as } x \to \infty$$

if $\exists\, r$ and C s.t.

$$|f(x)| < C\,|g(x)| \quad \forall\, x \in (r, \infty)$$

Suppose one could show

$$\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = M \geq 0$$

then by defn of these types of limits $\forall\, \varepsilon > 0$ $\exists\, r > 0$ s.t.

$$x \in (r,\infty) \Rightarrow \left| \frac{|f(x)|}{|g(x)|} - M \right| < \varepsilon$$

$$\Rightarrow -\varepsilon < \frac{|f(x)|}{|g(x)|} - M < \varepsilon$$

$$\Rightarrow M - \varepsilon < \frac{|f(x)|}{|g(x)|} < M + \varepsilon$$

$$\Rightarrow \frac{|f(x)|}{|g(x)|} < M + \varepsilon$$

$$\Rightarrow |f(x)| < (M+\varepsilon)\,|g(x)|$$

Since $M+\varepsilon$ is a constant for a given specific ε we would have what is in the box on the previous page. So it is enough to show

$$\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = \text{Real Number}$$

For a given $K > 0$

$$f(x) = \ln(x) \quad \text{\&} \quad g(x) = x^r$$

I don't want to worry about differentiating $|f(x)|$ & $|g(x)|$ as I will use L'Hospital's Rule. I can assume $r > 1$ so that x is larger than 1. Then

$$\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = \lim_{x \to \infty} \frac{|\ln(x)|}{|x^K|} = \lim_{x \to \infty} \frac{\ln(x)}{x^K} = \frac{+\infty}{+\infty}$$

using L'Hospital's rule then

$$\lim_{x \to \infty} \frac{\ln(x)}{x^r} = \lim_{x \to \infty} \frac{\frac{1}{x}}{kx^{k-1}} = \lim_{x \to \infty} \frac{1}{K} \frac{1}{x^K} = 0$$