M235 Calculus I: *Day 1*

The Language of Mathematics: Some Important Symbols, Notation, and Basic Proof Concepts

Basic Mathematical Shorthand:

$\exists$	means	"there exists" or "there is".
$\forall$	means	"for all" or "for every".
$\therefore$	means	"therefore" or "thus".
s.t.	means	"such that".
$\mathcal{S}$	means	"Suppose that"
$\in$	means	"an element of" or "in"
$\mathbb{R}$	represents	the set of real numbers $(-\infty,\infty)$
$\mathbb{N}$	represents	the set of natural numbers (1,2,3,…)
∞	represents	infinity (∞ is NOT a real number)
$\rightarrow$	means	"approaches" or "gets close to" or "goes to"
$\Rightarrow$	means	"implies"

Mathematical Statements:

I. The following are equivalent:

i.)	If A ,	then	B.	(If A is true then B is true).
ii.)	A	$\Rightarrow$	B.	(A implies B).
iii.)	B	$\Leftarrow$	A.	(B is implied by A).
iv.)	B	if	A.	(B is true if A is true).

In each of the above, "A" represents the *hypothesis*, and "B" represents the *conclusion.* In ***proving*** a statement like the above, one must show that the conclusion is a consequence of the hypothesis. That is, the hypothesis is given as a fact (assume the hypothesis), and we must show *through a series of logical and justifiable steps*, that the conclusion follows from the given hypothesis. ***Note:*** A correctly written mathematical proof **does not** begin with the conclusion and work backwards toward the hypothesis. However, this backwards analysis can be a useful process to see how to then properly write the proof, beginning with the hypothesis and ending up with the conclusion.

II. The following are equivalent:

i.)	A	if and only if	B.
ii.)	A	iff	B.
iii.)	A	$\Leftrightarrow$	B.
iv.)	A $\Rightarrow$ B	and	B $\Rightarrow$ A.

To prove an "if and only if" statement like those above, one must give **two** mathematical proofs. First you need to prove that A $\Rightarrow$ B (A is the hypothesis and B is the conclusion), and then you also need to prove that B $\Rightarrow$ A (B is the hypothesis and A is the conclusion).

III. Methods of proof: There are several ways to prove a statement of the form $A \Rightarrow B$ *(i.e., If A is true then B is true)* that we will encounter in the course of this class:

- **Direct Proof:** Start by assuming A is true and *through a series of logical and justifiable steps,* end by concluding that B must then necessarily be true.
- **Proof by Contradiction:** Start by assuming A is true ***and*** B is ***not*** true. Then, *through a series of logical and justifiable steps,* reach a contradiction to a known fact (for example, show A is not true or show 0=1 or $x^2 < 0$ or something like that). Thus you have reached a contradiction and so your assumption that "B is ***not*** true" is ***not*** true, and $\therefore$ B must then be true!
- **Proof by Contrapositive:** Prove Not B $\Rightarrow$ Not A. So start with the opposite of the conclusion and *through a series of logical and justifiable steps,* end by showing that the opposite of the hypothesis must hold. This is equivalent to proving $A \Rightarrow B$.
- **Proof by Induction:** A three step proof which is useful when trying to prove that some statement involving *n* holds for all natural numbers *n* *(i.e., for n=1,2,3,…).* The steps are
  1) First prove the statement for a simple or trivial case (e.g., when *n=1* or *n=2).*
  2) Next assume the statement holds when *n* equals some natural number, call it *k.* This is called the *Induction Hypothesis.*
  3) Now prove that the *Induction Hypothesis* $\Rightarrow$ the statement is true when *n=k+1.* This usually involves manipulating the *Induction Hypothesis* and the case from step 1) using algebra or inequalities to arrive at the *n=k+1* case.

IV. Common Misconception in attempts at Mathematical Proof of $A \Rightarrow B$:
In many cases you may need to work backwards a little to figure out how A and B are related. That is, you start with B and manipulate it to get back to A. THIS IS ***NOT*** A MATHEMATICAL PROOF OF $A \Rightarrow B$!!!
Such a "backwards" procedure is useful in figuring out the "tricky" step, but this is then simply your scratchwork and you must then write a correct proof of $A \Rightarrow B$ with your scratchwork as a reverse blueprint for how to then write up your proof ***correctly*** by *starting with* A and then *ending up with* B instead of the other way around as in your scratchwork. We will mostly use Direct Proofs and Proofs by Contradiction in this course. The following shows how to declare the start of your correct mathematical proof of a statement for these two types of proofs.

For a Direct Proof of $A \Rightarrow B$:

Proof: *Let A be true.*
Then …
.
. *(Here is the series of logical*
. *and justified steps.)*
.
Thus, B is true.

For a Proof by Contradiction of $A \Rightarrow B$:

Proof: *Let A be true. Suppose that B is not true.*
Then B is false and so…
.
. *(Here is the series of*
. *logical and justified steps*
. *which produce a contradiction*
. *to some known fact.)*
.
Thus we have a contradiction and so we can conclude that B must be true.