

Given: 1) $\lim_{x \rightarrow c} f(x) = L$

2) $\lim_{x \rightarrow c} g(x) = L$

3) $\exists \beta > 0$ s.t. $0 < |x - c| < \beta \Rightarrow f(x) \leq h(x) \leq g(x)$

Prove: $\lim_{x \rightarrow c} h(x) = L$

PROOF:

Let $\varepsilon > 0$ choose $\delta = \min(\beta, \delta_1, \delta_2)$ where δ_1 and δ_2 are defined below using givens 1) and 2). Since $\varepsilon > 0$ and as we were given $\lim_{x \rightarrow c} f(x) = L$ there exists **by**

the definition of limit $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon. \quad (0.1)$$

Similarly since $\varepsilon > 0$ and as we were given $\lim_{x \rightarrow c} g(x) = L$ there exists **by the definition**

of limit $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - L| < \varepsilon \quad (0.2)$$

Now as $\delta = \min(\beta, \delta_1, \delta_2)$ then by definition of *min*

$$\delta \leq \delta_1 \quad \& \quad \delta \leq \delta_2 \quad \& \quad \delta \leq \beta,$$

and using (0.1), (0.2), and given 3 we have also

$$\begin{aligned} 0 < |x - c| < \delta &\Rightarrow |f(x) - L| < \varepsilon \\ &\Rightarrow L - \varepsilon < f(x) < L + \varepsilon \end{aligned} \quad (0.3)$$

and

$$\begin{aligned} 0 < |x - c| < \delta &\Rightarrow |g(x) - L| < \varepsilon \\ &\Rightarrow L - \varepsilon < g(x) < L + \varepsilon \end{aligned} \quad (0.4)$$

and

$$0 < |x - c| < \delta \Rightarrow f(x) \leq h(x) \leq g(x). \quad (0.5)$$

Now given $0 < |x - c| < \delta$

$$\begin{aligned} &\Rightarrow f(x) \leq h(x) \leq g(x) && \text{using (0.5)} \\ &\Rightarrow f(x) \leq h(x) \leq g(x) < L + \varepsilon && \text{using (0.4)} \\ &\Rightarrow L - \varepsilon < f(x) \leq h(x) \leq g(x) < L + \varepsilon && \text{using (0.3)} \\ &\Rightarrow L - \varepsilon < h(x) < L + \varepsilon \\ &\Rightarrow |h(x) - L| < \varepsilon \end{aligned}$$

