1.

Given: 1) $\lim_{x\to c} f(x)=L$ 2) $\lim_{x\to c} g(x)=M$ 3) $h(x)=f(x)+g(x)$	**Prove:** $\lim_{x\to c} h(x)=L+M$

PROOF:

Let $\varepsilon>0$ choose $\delta=\min(\delta_1,\delta_2)$ where δ_1 and δ_2 are defined below using givens 1) and 2). Since $\varepsilon>0$ then $\frac{1}{2}\varepsilon>0$ and as we were given $\lim_{x\to c} f(x)=L$ there exists **<u>by the definition of limit</u>** $\delta_1>0$ such that

$$0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\frac{1}{2}\varepsilon. \qquad (1.1)$$

Similarly since $\frac{1}{2}\varepsilon>0$ and as we were given $\lim_{x\to c} g(x)=M$ there exists **<u>by the definition of limit</u>** $\delta_2>0$ such that

$$0<|x-c|<\delta_2 \Rightarrow |g(x)-M|<\frac{1}{2}\varepsilon \qquad (1.2)$$

Now as $\delta=\min(\delta_1,\delta_2)$ then by definition of *min*

$$\delta\le\delta_1 \quad \& \quad \delta\le\delta_2 \quad ,$$

and using(1.1) and (1.2) we have also

$$0<|x-c|<\delta \Rightarrow |f(x)-L|<\frac{1}{2}\varepsilon, \qquad (1.3)$$

and

$$0<|x-c|<\delta \Rightarrow |g(x)-M|<\frac{1}{2}\varepsilon. \qquad (1.4)$$

Now using (1.3) and (1.4) given $0<|x-c|<\delta$

$$\Rightarrow |f(x)-L|+|g(x)-M|<\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon$$

$$\Rightarrow |f(x)-L+g(x)-M|<\varepsilon \qquad \text{as } |a+b|<|a|+|b|$$

$$\Rightarrow |f(x)+g(x)-L-M|<\varepsilon$$

$$\Rightarrow |f(x)+g(x)-(L+M)|<\varepsilon$$

$$\Rightarrow |h(x)-(L+M)|<\varepsilon \qquad (\text{ using third given })$$

□

2.

Given: 1) $\lim_{x \to c} f(x) = L$
2) $k \in \mathbb{R} \neq 0$ *(constant)*
3) $h(x) = kf(x)$

Prove: $\lim_{x \to c} h(x) = kL$

PROOF:

Let $\varepsilon > 0$ the choice of δ is defined below using given 1). Since $\varepsilon > 0$ then $\frac{\varepsilon}{|k|} > 0$ (note by second given $k \neq 0$) and as we were given $\lim_{x \to c} f(x) = L$ there exists **by the definition of limit** $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{|k|}$$

$$\Rightarrow |k|\,|f(x) - L| < \varepsilon$$

$$\Rightarrow |kf(x) - kL| < \varepsilon \qquad \text{as } |a||b| = |ab|$$

$$\Rightarrow |h(x) - kL| < \varepsilon \qquad (\text{ using third given })$$

so

$$0 < |x - c| < \delta \Rightarrow |h(x) - L| < \varepsilon$$

□

Given: 1) $k \in \mathbb{R}$ (constant)
2) $h(x) = k$

Prove: $\lim_{x \to c} h(x) = k$

PROOF:

Let $\varepsilon > 0$

$$\Rightarrow |0| < \varepsilon$$

$$\Rightarrow |k - k| < \varepsilon$$

$$\Rightarrow |h(x) - k| < \varepsilon \quad (\text{ using third given })$$

so for ANY $\delta > 0$ (pick your favorite positive number),

$$0 < |x - c| < \delta \Rightarrow |h(x) - k| < \varepsilon$$

□

So the statement at the top is valid even when $k = 0$ as $\lim_{x \to c} 0 f(x) = \lim_{x \to c} 0 = 0 = 0L$

3.

Given: 1) $\lim_{x\to c} f(x) = L$ 2) $\lim_{x\to c} g(x) = M$ 3) $h(x) = f(x)g(x)$	**Prove:** $\lim_{x\to c} h(x) = LM$

PROOF:

Let $\varepsilon > 0$ choose $\delta = \min(\delta_1, \delta_2, \delta_3)$ where δ_1, δ_2, and δ_3 are defined below using givens 1) and 2). Since we are given $\lim_{x\to c} f(x) = L$ for $1 > 0$ there exists **<u>by the definition of limit</u>** $\delta_1 > 0$ such that

$$
\begin{aligned}
0 < |x - c| < \delta_1 \;&\Rightarrow |f(x) - L| < 1 \\
&\Rightarrow -1 < f(x) - L < 1 \\
&\Rightarrow L - 1 < f(x) < L + 1 \\
&\Rightarrow -|L| - 1 < f(x) < |L| + 1 \qquad \text{as } -|L| \le L \le |L| \\
&\Rightarrow -(|L| + 1) < f(x) < |L| + 1 \\
&\Rightarrow |f(x)| < |L| + 1,
\end{aligned}
$$

so

$$0 < |x - c| < \delta_1 \;\Rightarrow |f(x)| < |L| + 1. \tag{1.5}$$

Since $\varepsilon > 0$ then $\dfrac{\frac{1}{2}\varepsilon}{|L|+1} > 0$ and as we were given $\lim_{x\to c} g(x) = M$ there exists **<u>by the definition of limit</u>** $\delta_2 > 0$ such that

$$
\begin{aligned}
0 < |x - c| < \delta_2 \;&\Rightarrow |g(x) - M| < \frac{\frac{1}{2}\varepsilon}{|L|+1} \\
&\Rightarrow (|L| + 1)|g(x) - M| < \frac{1}{2}\varepsilon
\end{aligned}
\tag{1.6}
$$

Similarly since $\varepsilon > 0$ then $\dfrac{\frac{1}{2}\varepsilon}{|M|+1} > 0$ and as we were given $\lim_{x\to c} f(x) = L$ there exists **<u>by the definition of limit</u>** $\delta_3 > 0$ such that

$$
\begin{aligned}
0 < |x - c| < \delta_3 \;&\Rightarrow |f(x) - L| < \frac{\frac{1}{2}\varepsilon}{|M|+1}. \\
&\Rightarrow (|M| + 1)|f(x) - L| < \frac{1}{2}\varepsilon
\end{aligned}
\tag{1.7}
$$

Now as $\delta = \min(\delta_1, \delta_2, \delta_3)$ then by definition of *min*

$$\delta \le \delta_1 \quad \& \quad \delta \le \delta_2 \quad \& \quad \delta \le \delta_3,$$

and using (1.5),(1.6), and (1.7) we have also

$$0<|x-c|<\delta \ \Rightarrow |f(x)|<|L|+1\,, \tag{1.8}$$

$$0<|x-c|<\delta \ \Rightarrow (|L|+1)|g(x)-M|<\frac{1}{2}\varepsilon\,, \tag{1.9}$$

and

$$0<|x-c|<\delta \Rightarrow (|M|+1)|f(x)-L|<\frac{1}{2}\varepsilon\,. \tag{1.10}$$

Now using (1.9) and (1.10) given $0<|x-c|<\delta$

$$\Rightarrow (|M|+1)|f(x)-L| \ + \ (|L|+1)|g(x)-M|< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$

$$\Rightarrow |M|\ |f(x)-L| \ + \ (|L|+1)|g(x)-M|< \varepsilon \qquad \text{as } |M|\le|M|+1$$

This further implies using (1.8)

$$\Rightarrow |M|\ |f(x)-L| \ + \ |f(x)||g(x)-M|< \varepsilon$$

$$\Rightarrow \ |Mf(x)-ML| \ + \ |f(x)g(x)-f(x)M|< \varepsilon \qquad \text{as } |a||b\pm c|=|ac\pm ab|$$

$$\Rightarrow \ |Mf(x)-ML + \ f(x)g(x)-f(x)M|< \varepsilon \qquad \text{as } |a+b|\le|a|+|b|$$

$$\Rightarrow \ |f(x)g(x)-ML|< \varepsilon$$

$$\Rightarrow \ |h(x)-ML|< \varepsilon \qquad \text{using given 3)}$$

□

4.

Given: 1) $\lim_{x\to c} g(x) = M$ 2) $M \neq 0$ 3) $h(x) = \dfrac{1}{g(x)}$	**Prove:** $\lim_{x\to c} h(x) = \dfrac{1}{M}$

PROOF:

Let $\varepsilon > 0$ choose $\delta = \min(\delta_1, \delta_2)$ where δ_1 and δ_2 are defined below using givens 1) and 2). Since we are given $\lim_{x\to c} g(x) = M$ for $\dfrac{|M|}{2} > 0$ there exists **<u>by the definition of limit</u>** $\delta_1 > 0$ such that

$$0 < |x - c| < \delta_1 \;\Rightarrow |g(x) - M| < \frac{|M|}{2}$$

$$\Rightarrow \big||g(x)| - |M|\big| < \frac{|M|}{2} \qquad \text{as} \quad \big||a| - |b|\big| \le |a - b|$$

$$\Rightarrow -\frac{|M|}{2} < |g(x)| - |M| < \frac{|M|}{2}$$

$$\Rightarrow |M| - \frac{|M|}{2} < |g(x)| < |M| + \frac{|M|}{2}$$

$$\Rightarrow \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2}$$

$$\Rightarrow \frac{|M|}{2} < |g(x)|$$

$$\Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|} \qquad \text{if } a, b > 0 \text{ and } a < b \text{ then } \frac{1}{b} < \frac{1}{a}$$

so

$$0 < |x - c| < \delta_1 \;\Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|}. \qquad (1.11)$$

Since $\varepsilon > 0$ then $\dfrac{|M|^2 \varepsilon}{2} > 0$ (recall second given $M \neq 0$) and as we were given $\lim_{x\to c} g(x) = M$ there exists **<u>by the definition of limit</u>** $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \;\Rightarrow |g(x) - M| < \frac{|M|^2 \varepsilon}{2}$$

$$\Rightarrow \frac{2}{|M|^2} |g(x) - M| < \varepsilon \qquad (1.12)$$

$$\Rightarrow \frac{2}{|M|} \frac{1}{|M|} |g(x) - M| < \varepsilon$$

.

Now as $\delta = \min(\delta_1, \delta_2)$ then by definition of *min*

$$\delta \le \delta_1 \quad \& \quad \delta \le \delta_2 \quad ,$$

and using (1.11) and (1.12) we have also

$$0 < |x-c| < \delta \Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|}, \tag{1.13}$$

and

$$0 < |x-c| < \delta \Rightarrow \frac{2}{|M|}\frac{1}{|M|}|g(x)-M| < \varepsilon . \tag{1.14}$$

Now using (1.13) and (1.14) given $0 < |x-c| < \delta$

$$\Rightarrow \frac{2}{|M|}\frac{1}{|M|}|g(x)-M| < \varepsilon$$

$$\Rightarrow \frac{1}{|g(x)|}\frac{1}{|M|}|g(x)-M| < \varepsilon \qquad \text{as } \frac{1}{|g(x)|} < \frac{2}{|M|}$$

$$\Rightarrow \frac{1}{|g(x)|}\frac{1}{|M|}|M-g(x)| < \varepsilon \qquad \text{as } |a-b| = |b-a|$$

$$\Rightarrow \left|\frac{M-g(x)}{g(x)M}\right| < \varepsilon \qquad \text{as } |a||b| = |ab|$$

$$\Rightarrow \left|\frac{M}{g(x)M} - \frac{g(x)}{g(x)M}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \varepsilon$$

$$\Rightarrow \left|h(x) - \frac{1}{M}\right| < \varepsilon \qquad \text{using given 3)}$$

$\square$