Math/Phys 266E
Homework #4

Assigned: Th Apr. 5 *Due: Th Apr. 12*

1. **(75 points)** For the displacement in Spencer, p.89, #7: Compute:
(a) $\text{grad}\,\mathbf{u}(\mathbf{x},t)$, the displacement gradient tensor.
(b) $\mathbf{F} = \mathbf{I} + \text{grad}\mathbf{u}$. *Note that for the infinitesimal scenario, we neglect the distinction between* $\mathbf{x}$ *and* $\mathbf{X}$.
(c) The infinitesimal strain tensor $\mathbf{E} = \frac{1}{2}(\text{grad}\mathbf{u} + (\text{grad}\mathbf{u})^T)$.
(d) $(\text{grad}\mathbf{u})(\text{grad}\mathbf{u})^T$.
(e) $(\text{grad}\mathbf{u})^T(\text{grad}\mathbf{u})$. NOTE: (d) and (e) are different!
(f) $\boldsymbol{\gamma} = \mathbf{E} + \frac{1}{2}(\text{grad}\mathbf{u})^T(\text{grad}\mathbf{u})$. NOTE: The last term is the *geometric nonlinearity*!
(g) Use components of $\boldsymbol{\gamma}$ to determine the angle after deformation of two line elements originally in the $\mathbf{E}_1$ and $\mathbf{E}_2$ directions.
(h) Use components of $\mathbf{E}$ to determine an approximation to the angle after deformation of two line elements originally in the $\mathbf{E}_1$ and $\mathbf{E}_2$ directions. Compare with the previous part.
(i) Use components of $\boldsymbol{\gamma}$ to determine a formula for the angle after deformation at any (x_1, x_2, x_3) of two line elements originally in the $\mathbf{E}_1$ and $\mathbf{E}_3$ directions.
(j) Use components of $\mathbf{E}$ to determine an approximate formula for the angle at any (x_1, x_2, x_3) after deformation of two line elements originally in the $\mathbf{E}_1$ and $\mathbf{E}_3$ directions. Compare with the previous part. Note the lack of nonlinearity here in both $\mathbf{x}$ and ψ compared with the previous part.
(k) Show that the eigenvalues of $\mathbf{E}$ are $\lambda_1 = 0$ and $\lambda_2, \lambda_3 = \pm\frac{1}{2}\psi(x_1^2 + x_2^2)^{1/2}$. (So one of the principal components of infinitesimal strain is always zero!)
(l) Show that the eigenvector ($\mathbf{v}$) of $\mathbf{E}$ corresponding to $\lambda_1 = 0$ is $x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. (Use $(\mathbf{E} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, that is, let $\mathbf{v}$ be your eigenvector since we have components of $\mathbf{x}$ in $\mathbf{E}$).
(m) Show that the eigenvector ($\mathbf{v}$) of $\mathbf{E}$ corresponding to $\lambda_2 = \frac{1}{2}\psi(x_1^2 + x_2^2)^{1/2}$ is $-x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + (x_1^2 + x_2^2)^{1/2}\mathbf{e}_3$. Normalize this eigenvector and call it $\mathbf{m}$.
(n) Evaluate λ_2 and $\mathbf{m}$ at the surface of the rod if the rod has outer radius *a*.
(o) Compute $\mathbf{m}\bullet\mathbf{E}_3$ to determine the angle between the principal direction $\mathbf{m}$ and the x_3-axis at the surface of the rod.
(p) Note that this final result indicates that if you give a circular cylindrical rod (like a piece of chalk) a small twist, a principal direction of strain occurs at a 45° to the x_3-axis, and so if you twist until the chalk breaks, you should see this 45° angle at the surface!

2. **(25 points)** Show each of the following (they're simple and quick!) where $\mathbf{F}, \mathbf{U}, \mathbf{V}, \mathbf{R}$ are the deformation Gradient tensor, right and left stretch tensors, and rotation tensor, respectively.
(a) Solve the Polar Decomposition formulas to give $\mathbf{V}$ in terms of $\mathbf{U}, \mathbf{R}$, and $\mathbf{R}^T$ only.
(b) Use (a) to show that $I_1(\mathbf{V}) = I_1(\mathbf{U})$.
(c) Use (a) to show that $I_2(\mathbf{V}) = I_2(\mathbf{U})$.
(d) Use (a) to show that $I_3(\mathbf{V}) = I_3(\mathbf{U}) = I_3(\mathbf{F})$.

Note: The invariants of the left and right stretch tensors are often written as i_1, i_2, i_3 in mechanics literature.

(e) In general, are the eigenvalues of $\mathbf{U}$ the same as the eigenvalues of $\mathbf{F}$? Why or why not?
In general, are the eigenvalues of $\mathbf{U}$ the same as the eigenvalues of $\mathbf{V}$? Why or why not?
Write the principal isotropic invariants of $\mathbf{U}$ in terms of its eigenvalues? Why is this valid?