1. Highlights of Important M266E Material

Position, Velocity, Acceleration:

$\mathbf{X}$: original/undeformed configuration position vector (material coordinates)

$\mathbf{x}$: current/deformed configuration position vector (spatial coordinates)

$\mathbf{x}(\mathbf{X})$: deformation field

$\mathbf{x}(\mathbf{X},t)$: motion

$\mathbf{u}=\mathbf{x}-\mathbf{X}$: displacement field

$\mathbf{x}=\mathbf{X}+\mathbf{u}$: alternate representation of deformation field in terms of displacement field

$\mathbf{u}(\mathbf{X},t)$: material description of displacement field

$\mathbf{u}(\mathbf{x},t)$: spatial description of displacement field

$\mathbf{v}(\mathbf{X},t)=\frac{\partial\mathbf{x}(\mathbf{X},t)}{\partial t}=\frac{\partial\mathbf{u}(\mathbf{X},t)}{\partial t}$: material description of velocity field

$\mathbf{v}(\mathbf{x},t)$: spatial description of velocity field

$\mathbf{a}(\mathbf{X},t)=\frac{\partial\mathbf{v}(\mathbf{X},t)}{\partial t}=\frac{\partial^2\mathbf{x}(\mathbf{X},t)}{\partial t^2}=\frac{\partial^2\mathbf{u}(\mathbf{X},t)}{\partial t^2}$: material description of acceleration field

$\mathbf{a}(\mathbf{x},t)=\frac{D\mathbf{v}}{Dt}=\dot{\mathbf{v}}=\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t}+(\text{grad}\mathbf{v})\mathbf{v}=\mathbf{v}'+(\mathbf{v}\otimes\nabla_{\mathbf{x}})\mathbf{v}$: spatial description of acceleration field in terms of spatial velocity field $\mathbf{v}(\mathbf{x},t)$ (Material Time Derivative)

Stretch, Deformation, and Strain Tensors:

$\mathbf{I}$: Identity tensor

$\mathbf{F}=\text{Grad}\mathbf{x}=\mathbf{x}\otimes\nabla_{\mathbf{X}}=\mathbf{I}+\text{Grad}\mathbf{u}(\mathbf{X},t)$: deformation gradient tensor (Jacobian)

$\mathbf{F}=\mathbf{R}\mathbf{U}=\mathbf{V}\mathbf{R}$: polar decomposition

$\mathbf{R}$: rotation tensor (orthogonal: $\mathbf{R}^T=\mathbf{R}^{-1}$)

$\mathbf{U}$: right stretch tensor (symmetric: $\mathbf{U}=\mathbf{U}^T$)

$\mathbf{V}$: left stretch tensor (symmetric: $\mathbf{V}=\mathbf{V}^T$)

$\mathbf{C}=\mathbf{F}^T\mathbf{F}=\mathbf{U}^2=\mathbf{I}+\text{Grad}\mathbf{u}+(\text{Grad}\mathbf{u})^T+(\text{Grad}\mathbf{u})^T(\text{Grad}\mathbf{u})$: right Cauchy-Green deformation tensor (symmetric: $\mathbf{C}=\mathbf{C}^T$)

$\mathbf{B}=\mathbf{F}\mathbf{F}^T=\mathbf{V}^2=\mathbf{I}+\text{Grad}\mathbf{u}+(\text{Grad}\mathbf{u})^T+(\text{Grad}\mathbf{u})(\text{Grad}\mathbf{u})^T$: left Cauchy-Green deformation tensor (symmetric: $\mathbf{B}=\mathbf{B}^T$)

$\gamma=\frac{1}{2}(\mathbf{C}-\mathbf{I})=\frac{1}{2}(\text{Grad}\mathbf{u}+(\text{Grad}\mathbf{u})^T+(\text{Grad}\mathbf{u})^T(\text{Grad}\mathbf{u}))$: Lagrangian strain tensor (symmetric: $\gamma=\gamma^T$)

$\eta=\frac{1}{2}(\mathbf{I}-\mathbf{B}^{-1})$: Eulerian strain tensor (symmetric: $\eta=\eta^T$)

$\mathbf{E}=\frac{1}{2}(\text{Grad}\mathbf{u}+(\text{Grad}\mathbf{u})^T)$: linearized (infinitesimal) strain tensor for Linear Elastic Solids (LES) (symmetric: $\mathbf{E}=\mathbf{E}^T$; In indicial (Cartesian) notation: $E_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$). No Material/spatial distinction.

$\mathbf{D}=\frac{1}{2}(\text{grad}\mathbf{v}+(\text{grad}\mathbf{v})^T)$: Rate of Deformation Tensor (fluids–no linearization here!!) (symmetric part of $\mathbf{L}$)

$\mathbf{W}=\frac{1}{2}(\text{grad}\mathbf{v}-(\text{grad}\mathbf{v})^T)$: Spin Tensor (fluids–skew part of $\mathbf{L}$)

$\mathbf{L}=\text{grad}\mathbf{v}=\mathbf{v}\otimes\nabla_{\mathbf{x}}$: Spatial Velocity Gradient

$\dot{\mathbf{F}}=\text{Grad}\mathbf{v}=\mathbf{v}\otimes\nabla_{\mathbf{X}}=\frac{D\mathbf{F}}{Dt}$: Material Velocity Gradient

$\text{Grad}\mathbf{w}=(\text{grad}\mathbf{w})\mathbf{F}$: Relationship between Material Gradients and spatial gradients of vectors in Material and spatial descriptions

Eigenvalues and Invariants:

$\lambda_1^2, \lambda_2^2, \lambda_3^2$: eigenvalues of $\mathbf{B} = \mathbf{V}^2$ and $\mathbf{C} = \mathbf{U}^2$
$\lambda_1, \lambda_2, \lambda_3$: principal stretches, eigenvalues of $\mathbf{V}$ and $\mathbf{U}$
$J = \det\mathbf{F} = \lambda_1\lambda_2\lambda_3$: Jacobian determinant
$I_1 = \mathrm{tr}\mathbf{C} = \mathrm{tr}\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$: 1^{st} principal isotropic invariant
$I_2 = \frac{1}{2}(I_1^2 - \mathrm{tr}(\mathbf{C}^2)) = \frac{1}{2}(I_1^2 - \mathrm{tr}(\mathbf{B}^2)) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$: 2^{nd} principal isotropic invariant
$I_3 = \det\mathbf{C} = \det\mathbf{B} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$: 3^{rd} principal isotropic invariant
$I_2 = I_3\mathrm{tr}(\mathbf{C}^{-1}) = I_3\mathrm{tr}(\mathbf{B}^{-1})$: alternate expression for 2^{nd} principal isotropic invariant
$i_1 = \mathrm{tr}\mathbf{U} = \mathrm{tr}\mathbf{V} = \lambda_1 + \lambda_2 + \lambda_3$: 1^{st} stretch tensor isotropic invariant
$i_2 = \frac{1}{2}(i_1^2 - \mathrm{tr}(\mathbf{U}^2)) = \frac{1}{2}(i_1^2 - \mathrm{tr}(\mathbf{V}^2)) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$: 2^{nd} stretch tensor isotropic invariant
$i_3 = \det\mathbf{V} = \det\mathbf{U} = J = \lambda_1\lambda_2\lambda_3$: 3^{rd} stretch tensor isotropic invariant

Incompressibility Constraints:

$J = \det\mathbf{F} = 1$: incompressibility constraint for nonlinearly elastic solids (NES)
$\mathrm{tr}\mathbf{E} = \mathrm{div}\mathbf{u} = 0$: incompressibility constraint for linearly elastic solids (LES)
$\mathrm{tr}\mathbf{D} = \mathrm{div}\mathbf{v} = 0$: incompressibility constraint for Newtonian viscous fluids (NVF)

Basic Material Models:

W: stored-energy density function per unit undeformed volume (material model)
$W(I_1, I_2, I_3)$ or $W(i_1, i_2, i_3)$: compressible isotropic material models for NES
$W(I_1, I_2)$ or $W(i_1, i_2)$: incompressible isotropic material models for NES
$W = \frac{\mu}{2}(I_1 - 3)$: neo-Hookean
$W = c_1(I_1 - 3) + c_2(I_2 - 3)$: Mooney-Rivlin
$W = \frac{\mu}{2}(\frac{I_2}{I_3} + 2I_3^{\frac{1}{2}} - 5)$: Special Blatz-Ko

Stretch in any Direction/Change in Angles:

$\lambda(\mathbf{M}) = \sqrt{\mathbf{M} \cdot (\mathbf{C}\mathbf{M})}$: stretch at $\mathbf{X}$ in direction with unit vector $\mathbf{M}$ in original configuration
$\lambda^{-1}(\mathbf{m}) = \sqrt{\mathbf{m} \cdot (\mathbf{B}^{-1}\mathbf{m})}$: inverse of stretch at $\mathbf{x}$ in unit vector $\mathbf{m}$ direction in deformed configuration
$\cos\theta = \frac{\mathbf{M}_1 \cdot (\mathbf{C}\mathbf{M}_2)}{\lambda(\mathbf{M}_1)\lambda(\mathbf{M}_2)}$: cosine of deformed angle between line elements (tangent vectors to curves at a point $\mathbf{x}$) that originally had unit vectors in the $\mathbf{M}_1$ and $\mathbf{M}_2$ directions in the original configuration
$\cos\theta = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{2\gamma_{12}}{\sqrt{1+2\gamma_{11}}\sqrt{1+2\gamma_{22}}}$: cosine of deformed angle between line elements originally in $\mathbf{E}_1$ and $\mathbf{E}_2$ directions in the original configuration
$\cos\theta \approx 2E_{12}$: linearized approximation to cosine of deformed angle between line elements originally in $\mathbf{E}_1$ and $\mathbf{E}_2$ directions in the original configuration (similar for $\mathbf{E}_1$ and $\mathbf{E}_3$ and $\mathbf{E}_2$ and $\mathbf{E}_3$ directions)
$\cos\Theta = \frac{\mathbf{m}_1 \cdot (\mathbf{B}^{-1}\mathbf{m}_2)}{\lambda^{-1}(\mathbf{m}_1)\lambda^{-1}(\mathbf{m}_2)}$: cosine of original angle between line elements that in the deformed configuration had unit vectors (tangent vectors to curves at a point $\mathbf{x}$) in the $\mathbf{m}_1$ and $\mathbf{m}_2$ directions.

Stress Tensors/Traction Vectors:

$\mathbf{T}$: Cauchy stress tensor (symmetric: $\mathbf{T} = \mathbf{T}^T$)
$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}$: nominal stress tensor (***not*** symmetric: $\mathbf{S} \neq \mathbf{S}^T$)
$\mathbf{S}^T = J\mathbf{T}\mathbf{F}^{-T}$: 1^{st} Piola-Kirchhoff stress tensor
$\mathbf{P} = \mathbf{S}\mathbf{F}^{-T} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$: 2^{nd} Piola-Kirchhoff stress tensor (symmetric: $\mathbf{P} = \mathbf{P}^T$)
$\mathbf{t}(\mathbf{x}) = \mathbf{T}^T\mathbf{n} = \mathbf{T}\mathbf{n}$: traction vector, unit normal $\mathbf{n}$ is to deformed configuration surface
$\mathbf{s}(\mathbf{X}) = \mathbf{S}^T\mathbf{N}$: traction vector, unit normal $\mathbf{N}$ is to undeformed configuration surface

Tensor Components and Representation in terms of Basis Tensors:

Nominal Stress: $\mathbf{S} = S_{ij}\mathbf{E}_i \otimes \mathbf{e}_j$ Components: $S_{ij} = \mathbf{E}_i \cdot \mathbf{S}\mathbf{e}_j = \mathbf{S}^T\mathbf{E}_i \cdot \mathbf{e}_j = \mathbf{s}_i \cdot \mathbf{e}_j$
Cauchy Stress: $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ Components: $T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{T}^T\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{t}_i \cdot \mathbf{e}_j$
Deformation Gradient: $\mathbf{F} = F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$ Components: $F_{ij} = \mathbf{e}_i \cdot \mathbf{F}\mathbf{E}_j$
Deformation Tensors: $\mathbf{B} = \mathbf{F}\mathbf{F}^T = B_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\mathbf{C} = \mathbf{F}^T\mathbf{F} = C_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$
Components: $\mathbf{B}_{ij} = \mathbf{e}_i \cdot \mathbf{B}\mathbf{e}_j$, $C_{ij} = \mathbf{E}_i \cdot \mathbf{C}\mathbf{E}_j$

Equilibrium Equations (in the absence of body forces) and Equations of Motion (including body forces) for a Continuum :

$\mathrm{div}\mathbf{T} = \mathbf{0}$ and $\mathrm{div}\mathbf{T} + \rho\mathbf{b} = \rho\mathbf{a}$
$\mathrm{Div}\mathbf{S} = \mathbf{0}$ and $\mathrm{Div}\mathbf{S} + \rho_0\mathbf{b}_0 = \rho_0\mathbf{a}$

Linear Elastic Solid (LES) Governing equations: (no $\mathbf{x}$ vs. $\mathbf{X}$ distinction)

(Note: Below, $\nabla^2\mathbf{u} = \mathrm{div}((\mathrm{grad}\mathbf{u})^T) = \Delta\mathbf{u}$, Δ : Laplacian operator)

Compressible

Constitutive Equation: $\mathbf{T} = \lambda(\mathrm{tr}\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$, $\lambda = \frac{2\mu}{1-2\nu}$, μ, ν : material constants.
Equations of Motion: $\mu\nabla^2\mathbf{u} + (\lambda + \mu)\mathrm{grad}(\mathrm{div}\mathbf{u}) + \rho_0\mathbf{b} = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}$.
Indicial Notation (i=1,2,3): $\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + \rho_0 b_i = \rho_0\frac{\partial^2 u_i}{\partial t^2}$ (implied summation on k from 1 to 3).

Incompressible:

Incompressibility Constraint: $\mathrm{tr}\mathbf{E} = \mathrm{div}\mathbf{u} = 0$.
Constitutive Equation: $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{E}$, $p : \mathrm{pressure,\ non-constant.}$
Equations of Motion: $\mu\nabla^2\mathbf{u} - \mathrm{grad}p + \rho_0\mathbf{b} = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}$.
Indicial Notation (i=1,2,3): $\mu u_{i,kk} + p_{,i} + \rho_0 b_i = \rho_0\frac{\partial^2 u_i}{\partial t^2}$ (implied summation on k from 1 to 3).

Incompressible Newtonian Viscous Fluid (NVF) Governing equations: (no linearizations!!) $\nabla^2\mathbf{v} = \mathrm{div}((\mathrm{grad}\mathbf{v})^T) = \Delta\mathbf{v}$, Δ : Laplacian operator)

Incompressibility Constraint: $\mathrm{tr}\mathbf{D} = \mathrm{div}\mathbf{v} = 0$
Constitutive Equation: $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$, $p : \mathrm{pressure,\ non-constant.}$
Equations of Motion: $\mu\nabla^2\mathbf{v} - \mathrm{grad}p + \rho\mathbf{b} = \rho_0\dot{\mathbf{v}}$, ($\mathbf{v}$: ***spatial field*** $\Rightarrow$ ***nonlinearity!***).
So, equivalently: $\mu\nabla^2\mathbf{v} - \mathrm{grad}p + \rho\mathbf{b} = \rho_0(\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t} + (\mathrm{grad}\mathbf{v})\mathbf{v})$.

Nonlinear Elastic Solid (NES) Constitutive Equations:

Compressible

$$\mathbf{T} = \frac{2}{J}(\frac{\partial W}{\partial I_3}I_3\mathbf{I} + (\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}I_1)\mathbf{B} - \frac{\partial W}{\partial I_2}\mathbf{B}^2)$$

$$\mathbf{T} = \frac{2}{J}(\frac{\partial W}{\partial I_2}I_2 + \frac{\partial W}{\partial I_3}I_3)\mathbf{I} + \frac{\partial W}{\partial I_1}\mathbf{B} - \frac{\partial W}{\partial I_2}I_3\mathbf{B}^{-1})$$

$$\mathbf{S} = 2(\frac{\partial W}{\partial I_3}I_3\mathbf{F}^{-1} + (\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2})\mathbf{F}^T - \frac{\partial W}{\partial I_2}\mathbf{C}\mathbf{F}^T)$$

$$\mathbf{S} = 2((\frac{\partial W}{\partial I_2}I_2 + \frac{\partial W}{\partial I_3}I_3)\mathbf{F}^{-1} + \frac{\partial W}{\partial I_1}\mathbf{F}^T - \frac{\partial W}{\partial I_2}I_3\mathbf{C}^{-1}\mathbf{F}^{-1})$$

Incompressible:

$$\text{Incompressibility Constraint}: \quad J \equiv \det\mathbf{F} = I_3^{\frac{1}{2}} = 1$$

$$\mathbf{T} = -p\mathbf{I} + 2\frac{\partial W}{\partial I_1}\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^{-1}, \qquad p(\mathbf{x}) : \text{unknown pressure}$$

$$\mathbf{T} = -p\mathbf{I} + 2(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2})\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^2, \qquad p(\mathbf{x}) : \text{unknown pressure}$$

$$\mathbf{S} = -p\mathbf{F}^{-1} + 2\frac{\partial W}{\partial I_1}\mathbf{F}^T - 2\frac{\partial W}{\partial I_2}\mathbf{C}^{-1}\mathbf{F}^{-1}, \qquad p(\mathbf{X}) : \text{unknown pressure}$$

$$\mathbf{S} = -p\mathbf{F}^{-1} + 2(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2})\mathbf{F}^T - 2\frac{\partial W}{\partial I_2}\mathbf{C}\mathbf{F}^T, \qquad p(\mathbf{X}) : \text{unknown pressure}$$

2. DEFORMATION GRADIENT TENSOR AND NES EQUILIBRIUM EQUATIONS: RECTANGULAR CARTESIAN COORDINATES

Let $\{X_1, X_2, X_3\} = \{X, Y, Z\}$ represent rectangular Cartesian Lagrangean (material) coordinates with orthonormal basis vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and origin $\mathbf{0}$, and let $\{x_1, x_2, x_3\} = \{x, y, z\}$ represent the corresponding rectangular Cartesian Eulerian (spatial) coordinates with orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and origin $\mathbf{o}$.

Consider the completely general deformation field

$$x = x(X, Y, Z), \quad y = y(X, Y, Z), \quad z = z(X, Y, Z).$$

The deformation gradient tensor referred to rectangular Cartesian coordinates is

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix},$$

The *nominal* stress tensor, $\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}$ and Cauchy stress tensor $\mathbf{T}$ have matrix representations (referred to each of their basis tensors)

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \qquad \text{and} \qquad [\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

For the symmetric Cauchy stress tensor, the coordinate form of $\text{div}\mathbf{T} = \mathbf{0}$ (where div denotes the divergence with respect to (x, y, z), the deformed coordinates (dependent variables)), gives the equilibrium equations, in the absence of body forces as

$$\begin{aligned} T_{11,_1} + T_{21,_2} + T_{31,_3} &= 0, \\ T_{12,_1} + T_{22,_2} + T_{32,_3} &= 0, \\ T_{13,_1} + T_{23,_2} + T_{33,_3} &= 0. \end{aligned}$$

The coordinate form of $\text{Div}\mathbf{S} = \mathbf{0}$ (where Div denotes the divergence with respect to (X, Y, Z), the undeformed coordinates (independent variables)), gives the equilibrium equations, in the absence of body forces as

$$\begin{aligned} S_{11,_1} + S_{21,_2} + S_{31,_3} &= 0, \\ S_{12,_1} + S_{22,_2} + S_{32,_3} &= 0, \\ S_{13,_1} + S_{23,_2} + S_{33,_3} &= 0. \end{aligned}$$

where, in the above and subsequently, a comma denotes appropriate partial differentiation. Recall that the Cauchy stress tensor is symmetric, while the nominal stress tensor is not!

3. DEFORMATION GRADIENT TENSOR AND NES EQUILIBRIUM EQUATIONS: CYLINDRICAL POLAR COORDINATES

Introduce the cylindrical coordinates

$$\{X_1, X_2, X_3\} = \{R\cos\Theta, R\sin\Theta, Z\}$$

and

$$\{x_1, x_2, x_3\} = \{r\cos\theta, r\sin\theta, z\}$$

with corresponding orthonormal basis vectors

$$\begin{aligned} \mathbf{E}_R &= \cos\Theta\mathbf{E}_1 + \sin\Theta\mathbf{E}_2, \\ \mathbf{E}_\Theta &= -\sin\Theta\mathbf{E}_1 + \cos\Theta\mathbf{E}_2, \\ \mathbf{E}_Z &= \mathbf{E}_3, \end{aligned}$$

and

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2,\\ \mathbf{e}_\theta &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,\\ \mathbf{e}_z &= \mathbf{e}_3,\end{aligned}$$

for $\mathbf{X}$ and $\mathbf{x}$ respectively.

Consider the completely general deformation field

$$r = r(R, \Theta, Z), \quad \theta = \theta(R, \Theta, Z), \quad z = z(R, \Theta, Z)$$

The deformation gradient tensor referred to cylindrical polar coordinates is

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix},$$

The *nominal* stress tensor, $\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}$ and Cauchy stress tensor $\mathbf{T}$ have matrix representations (referred to each of their basis tensors)

$$[\mathbf{S}] = \begin{bmatrix} S_{Rr} & S_{R\theta} & S_{Rz} \\ S_{\Theta r} & S_{\Theta\theta} & S_{\Theta z} \\ S_{Zr} & S_{Z\theta} & S_{Zz} \end{bmatrix}, \qquad \text{and} \qquad [\mathbf{T}] = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{bmatrix}$$

Recall that the Cauchy stress tensor is symmetric, while the nominal stress tensor is not!

The cylindrical polar coordinate form of div$\mathbf{T} = \mathbf{0}$ (where div denotes the divergence with respect to (r, θ, z), the deformed coordinates (dependent variables)), gives the equilibrium equations, in the absence of body forces as

$$\begin{aligned} T_{rr,_r} + \frac{1}{r}T_{r\theta,_\theta} + T_{rz,_z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) &= 0,\\ T_{r\theta,_r} + \frac{1}{r}T_{\theta\theta,_\theta} + T_{\theta z,_z} + \frac{2}{r}T_{r\theta} &= 0,\\ T_{rz,_r} + \frac{1}{r}T_{\theta z,_\theta} + T_{zz,_z} + \frac{1}{r}T_{rz} &= 0. \end{aligned}$$

The cylindrical polar coordinates form of Div$\mathbf{S} = \mathbf{0}$ (where Div denotes the divergence with respect to (R, Θ, Z), the undeformed coordinates (independent variables)), gives the equilibrium equations, in the absence of body forces as

$$S_{Rr,_R} - \theta_{,R}S_{R\theta} - \theta_{,Z}S_{Z\theta} + S_{Zr,_Z} + \frac{1}{R}\left[S_{Rr} - \theta_{,\Theta}S_{\Theta\theta} + S_{\Theta r,_\Theta}\right] = 0,$$

$$S_{R\theta,_R} + \theta_{,R}S_{Rr} + \theta_{,Z}S_{Zr} + S_{Z\theta,_Z} + \frac{1}{R}\left[S_{R\theta} + \theta_{,\Theta}S_{\Theta r} + S_{\Theta\theta,_\Theta}\right] = 0,$$

$$S_{Rz,_R} + S_{Zz,_Z} + \frac{1}{R}\left[S_{Rz} + S_{\Theta z,_\Theta}\right] = 0.$$

4. DEFORMATION GRADIENT TENSOR AND NES EQUILIBRIUM EQUATIONS: SPHERICAL POLAR COORDINATES

Introduce the spherical coordinates

$$\{X_1, X_2, X_3\} = \{R\sin\Phi\cos\Theta, R\sin\Phi\sin\Theta, R\cos\Phi\}$$

and

$$\{x_1, x_2, x_3\} = \{r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi\}$$

with corresponding orthonormal basis vectors

$$\begin{aligned}\mathbf{E}_R &= \sin\Phi\cos\Theta\mathbf{E}_1 + \sin\Phi\sin\Theta\mathbf{E}_2 + \cos\Phi\mathbf{E}_3,\\ \mathbf{E}_\Theta &= -\sin\Theta\mathbf{E}_1 + \cos\Theta\mathbf{E}_2,\\ \mathbf{E}_\Phi &= \cos\Phi\cos\Theta\mathbf{E}_1 + \cos\Phi\sin\Theta\mathbf{E}_2 - \sin\Phi\mathbf{E}_3,\end{aligned}$$

and

$$\begin{aligned}\mathbf{e}_r &= \sin\varphi\cos\theta\mathbf{e}_1 + \sin\varphi\sin\theta\mathbf{e}_2 + \cos\varphi\mathbf{e}_3,\\ \mathbf{e}_\theta &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2,\\ \mathbf{e}_\varphi &= \cos\varphi\cos\theta\mathbf{e}_1 + \cos\varphi\sin\theta\mathbf{e}_2 - \sin\varphi\mathbf{e}_3,\end{aligned}$$

for $\mathbf{X}$ and $\mathbf{x}$ respectively.

Consider the completely general deformation field

$$r = r(R,\Theta,\Phi), \quad \theta = \theta(R,\Theta,\Phi), \quad \varphi = \varphi(R,\Theta,\Phi)$$

Note that in the above, r and R represent *spherical* radial variables (in the previous section, r and R represent *cylindrical* polar radial variables).

The deformation gradient tensor referred to spherical polar coordinates is

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R\sin\Phi}\frac{\partial r}{\partial\Theta} & \frac{1}{R}\frac{\partial r}{\partial\Phi} \\ r\sin\varphi\frac{\partial\theta}{\partial R} & \frac{r\sin\varphi}{R\sin\Phi}\frac{\partial\theta}{\partial\Theta} & \frac{r\sin\varphi}{R}\frac{\partial\theta}{\partial\Phi} \\ r\frac{\partial\varphi}{\partial R} & \frac{r}{R\sin\Phi}\frac{\partial\varphi}{\partial\Theta} & \frac{r}{R}\frac{\partial\varphi}{\partial\Phi} \end{bmatrix},$$

The *nominal* stress tensor, $\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}$ and Cauchy stress tensor $\mathbf{T}$ have matrix representations (referred to each of their basis tensors)

$$[\mathbf{S}] = \begin{bmatrix} S_{Rr} & S_{R\theta} & S_{R\varphi} \\ S_{\Theta r} & S_{\Theta\theta} & S_{\Theta\varphi} \\ S_{\Phi r} & S_{\Phi\theta} & S_{\Phi\varphi} \end{bmatrix} \qquad \text{and} \qquad [\mathbf{T}] = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\varphi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\varphi} \\ T_{\varphi r} & T_{\varphi\theta} & T_{\varphi\varphi} \end{bmatrix}.$$

Recall that the Cauchy stress tensor is symmetric, while the nominal stress tensor is not!

The spherical polar coordinate form of $\text{div}\mathbf{T} = \mathbf{0}$ (where div denotes the divergence with respect to (r,θ,φ), the deformed coordinates (dependent variables)), gives the equilibrium equations, in the absence of body forces as

$$T_{rr,r} + \frac{2}{r}T_{rr} + \frac{1}{r}T_{r\varphi,\varphi} + \frac{1}{r\sin\varphi}T_{r\theta,\theta} + \frac{\cot\varphi}{r}T_{r\varphi} - \frac{1}{r}(T_{\varphi\varphi} + T_{\theta\theta}) = 0,$$

$$T_{r\theta,r} + \frac{3}{r}T_{r\theta} + \frac{1}{r}T_{\theta\varphi,\varphi} + \frac{1}{r\sin\varphi}T_{\theta\theta,\theta} + \frac{2\cot\varphi}{r}T_{\theta\varphi} = 0,$$

$$T_{r\varphi,r} + \frac{3}{r}T_{r\varphi} + \frac{1}{r}T_{\varphi\varphi,\varphi} + \frac{1}{r\sin\varphi}T_{\theta\varphi,\theta} + \frac{\cot\varphi}{r}(T_{\varphi\varphi} - T_{\theta\theta}) = 0.$$

The spherical polar coordinates form of Div$\mathbf{S} = \mathbf{0}$ (where Div denotes the divergence with respect to (R, Θ, Φ), the undeformed coordinates (independent variables)), gives the equilibrium equations, in the absence of body forces as

$$\begin{gathered}
S_{Rr,R} - \sin\varphi\,\theta_{,R}S_{R\theta} - \varphi_{,R}S_{R\varphi} + \frac{1}{R}\left[2S_{Rr} + \cot\Phi S_{\Phi r} + S_{\Phi r,\Phi} - \sin\varphi\,\theta_{,\Phi}S_{\Phi\theta} - \varphi_{,\Phi}S_{\Phi\varphi}\right] \\
+\frac{1}{R\sin\Phi}\left[S_{\Theta r,\Theta} - \sin\varphi\,\theta_{,\Theta}S_{\Theta\theta} - \varphi_{,\Theta}S_{\Theta\varphi}\right] = 0,
\end{gathered}$$

$$\begin{gathered}
\sin\varphi\,\theta_{,R}S_{Rr} + S_{R\theta,R} + \cos\varphi\,\theta_{,R}S_{R\varphi} + \frac{1}{R}\left[2S_{R\theta} + \cot\Phi S_{\Phi\theta} + \sin\varphi\,\theta_{,\Phi}S_{\Phi r} + S_{\Phi\theta,\Phi} + \cos\varphi\,\theta_{,\Phi}S_{\Phi\varphi}\right] \\
+\frac{1}{R\sin\Phi}\left[\sin\varphi\,\theta_{,\Theta}S_{\Theta r} + S_{\Theta\theta,\Theta} + \cos\varphi\,\theta_{,\Theta}S_{\Theta\varphi}\right] = 0,
\end{gathered}$$

$$\begin{gathered}
\varphi_{,R}S_{Rr} - \cos\varphi\,\theta_{,R}S_{R\theta} + S_{R\varphi,R} + \frac{1}{R}\left[2S_{R\varphi} + \cot\Phi S_{\Phi\varphi} + \varphi_{,\Phi}S_{\Phi r} - \cos\varphi\,\theta_{,\Phi}S_{\Phi\theta} + S_{\Phi\varphi,\Phi}\right] \\
+\frac{1}{R\sin\Phi}\left[\varphi_{,\Theta}S_{\Theta r} - \cos\varphi\,\theta_{,\Theta}S_{\Theta\theta} + S_{\Theta\varphi,\Theta}\right] = 0.
\end{gathered}$$