Step by Step Method for Calculating a Taylor Polynomial Approximation of a Function Expanded About a Point a.

STEP 1: Find the degree n required for the Taylor Polynomial. This is typically $n=3$ or $n=4$.

STEP 2: Find and label $f(x)$, the function that is to be expanded, and also the real number a the point that Taylor polynomial will be centered around.

STEP 3: The basic form of the Taylor polynomial approximating $f(x)$ for values x close to a is

$$f(x) \approx T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n \tag{1.1}$$

where $c_0, c_1, c_2, \cdots c_n$ are constants that must be determined. Write down the expression in (1.1) for the function $f(x)$ and the numerical value of a from step 2, and the natural number n from step 1. Keep the constants $c_0, c_1, c_2, \cdots c_n$ as they are (unknown) at this point.

STEP 4: The "real work" is about to start. Get ready to take some derivatives, find, and write out, expressions

$$f'(x),\ f''(x),\ f'''(x), \cdots,\ f^{(n)}(x),\ f^{(n+1)}(x). \tag{1.2}$$

This is going to require lots of MATH 235; product rule, quotient rule, chain rule, or simply knowing how to differentiate basic functions! The expression for $f^{(n+1)}(x)$ will only be needed if you need to bound the error in the Taylor polynomial (see step 8!)

STEP 5: Next evaluate $f(x),\ f'(x),\ f''(x),\ f'''(x), \cdots,\ f^{(n)}(x)$ at the numerical value $x=a$ creating the numbers $f(a),\ f'(a),\ f''(a),\ f'''(a), \cdots,\ f^{(n)}(a)$.

STEP 6: Now we determine the constants $c_0, c_1, c_2, \cdots c_n$ which we left unknown back in step 3, using the values calculated in step 5 by setting,

$$\begin{aligned} c_0 &= f(a), \\ c_1 &= f'(a), \\ c_2 &= \frac{f''(a)}{2!}, \\ c_3 &= \frac{f'''(a)}{3!}, \cdots, \\ c_n &= \frac{f^{(n)}(a)}{n!}. \end{aligned} \tag{1.3}$$

STEP 7: Substitute the numbers from step 6 in (1.3) into the expression (1.1), left in step 3, which is a n^{th} degree polynomial denoted $T_n(x)$, but don't expand it out just leave it in powers of $x-a$.

If you are asked to determine the entire series, then you need to look for a pattern in the first few coefficients found in step 6 and then generalize it to all terms in the series.

STEP 8: If you are asked to *bound the error* when the polynomial, T_n, from step 7 is used to approximate f at some value $x=b$, then go back to step 4 and write down your expression for $f^{(n+1)}(x)$. Find M, which is the largest value that $\left|f^{(n+1)}(x)\right|$ can take for any x in the interval determined by a and b. That is,

$$\left|f^{(n+1)}(x)\right| \le M \quad \forall\, x \in [a,b], \text{ (or if } b<a, \forall\, x \in [b,a])$$

Of course, that might take some effort, as essentially it is a maximization problem on a closed interval (this is similar to what we did with error bounds in numerical integration!). Once M is determined, then the absolute error in using the approximate value $T_n(b)$ instead of the actual value $f(b)$ is such that

$$\left|f(b)-T_n(b)\right| \le \frac{M}{(n+1)!}|b-a|^{n+1} \tag{1.4}$$

Note that in the above, you want to determine the expression on the right. You know *a*,*b*, and *n* as givens from the problem. The main task is to find *M.*

A related problem (like some we've done with numerical integration and alternating series) is to determine the smallest degree *n* of a Taylor polynomial needed to approximate the value of a function f at some value $x=b$ which is nearby the value $x=a$, where *a* is the value you've expanded about in your Taylor polynomial, such that the error in the approximation will be *guaranteed* to be less than some given value. Then we want to find the first value of n such that

$$\frac{M}{(n+1)!}|b-a|^{n+1} < \textit{error tolerance}$$

and then, by (1.4), and the transitive law, we'll have

$$\left|f(b)-T_n(b)\right| < \textit{error tolerance}.$$