Integration by Partial Fractions

THIS IS ALGEBRA UNTIL YOU ACTUALLY START TO INTEGRATE

STEP 1) You will have something in the form of a polynomial over a polynomial. The first step is to factor the denominator if at all possible:

$$y = \frac{ax+b}{cx^2+dx+e} = \frac{ax+b}{(qx+p)(rx+n)}$$

EXAMPLE

$$y = \frac{6x+7}{x^2-5x+6} = \frac{6x+7}{(x-2)(x-3)}$$

It may be necessary to factor out some constants from the polynomials, do so if at all possible.

STEP 2) It is a property of Algebra that this kind of "rational polynomial" can be altered so that it looks like:

$$y = \frac{A}{(qx+p)} + \frac{B}{(rx+n)}$$

EXAMPLE

$$y = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Note that the ***"partial fraction denominators"*** *are simply identified after factoring the denominator of the example. A and B are called* ***"undetermined coefficients".*** That is, they are coefficients that are not yet determined!! Now, to find *A* and *B:*

STEP 3) Since *y* is still equal to *y*, we can set them equal:

$$\frac{A}{(qx+p)} + \frac{B}{(rx+n)} = \frac{ax+b}{(qx+p)(rx+n)}$$

EXAMPLE

$$\frac{A}{(x-2)} + \frac{B}{(x-3)} = \frac{6x-7}{(x-2)(x-3)}$$

From this point on, it's not difficult at all, but there is algebra. You may already have noticed that high school algebra is the only **true arch-nemesis** to the calculus student…

STEP 4) Basic, 8^{th} grade algebra here… cross-multiplying.

$$A(rx+n) + B(qx+p) = ax+b$$

EXAMPLE

$$A(x-3) + B(x-2) = 6x-7$$

STEP 5) Since the equation in step 4 must hold **for all** values of x, we pick two CLEVER values of x to make our job of determining A and B as easy as possible.

EXAMPLE

$$x = \frac{-n}{r} \quad \& \quad x = \frac{-p}{q} \qquad\qquad x = 3 \quad \& \quad x = 2$$

Note that A and B are coefficients, what's more, they are numbers that you will usually recognize if this is a text book problem.

STEP 6) Now it is easy to solve for coefficients A and B as in our example.

$$A(x-3) + B(x-2) = 6x - 7$$

$$x = 3 \Rightarrow \quad A(0) + B(1) = 18 - 7 \;\Rightarrow\; B = 11$$

$$x = 2 \Rightarrow \quad A(-1) + B(0) = 12 - 7 \;\Rightarrow\; A = -5$$

Note that you know A and B are coefficients, you just don't know what they are equal to; they are undetermined. Because they are undetermined coefficients we cleverly name them: "undetermined coefficients." After Step 6), they are now determined. What remains to do is the integration!

STEP 7) So now we have converted our original problem into two fractions instead of one:

$$\frac{6x+7}{x^2-5x+6} = \frac{6x+7}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} = \frac{-5}{(x-2)} + \frac{11}{(x-3)}$$

STEP 8) Finally we get to do calculus. Separate y into its two separate parts to integrate:

$$\int \frac{6x+7}{x^2-5x+6}\,dx = \int \left[\frac{-5}{(x-2)} + \frac{11}{(x-3)} \right] dx = \int \frac{-5}{(x-2)}\,dx + \int \frac{11}{(x-3)}\,dx$$

$$= -5 \ln|x-2| + 11 \ln|x-3| + C$$

There, after all that, one step of calculus. Circle it and hand it in… You are DONE!

SPECIAL CASES!!!!

1] Anytime that the **degree of the numerator is larger than the degree of the denominator…** you'll need to use long division, or synthetic division. This will give a polynomial for you to integrate plus a *remainder* in the form of yet another polynomial fraction. *(Disappointing, but life is like a box of chocolates….)*

SIMPLE EXAMPLE:

$$y = \int \frac{x^2+9}{x}\ dx$$

$$\text{denominator}\overline{\big)\text{numerator}}^{\text{answer}} + \frac{\text{remainder}}{\text{denominator}}$$

$$x\overline{\big)x^2+9}^{\,x} = x + \frac{9}{x}$$

$$\underline{-x^2 - 0}$$

$$+9$$

(this is pretty obvious, but hopefully you get the idea for "complicated" divisions). So, then

$$y = \int \frac{x^2+9}{x}\ dx = \int \left[x + \frac{9}{x} \right] dx = \int x\,dx + \int \frac{9}{x} dx$$

$$= \frac{1}{2}x^2 + 9\ln|x| + C$$

"WARNEING " (*I know bad humor*) not all remainders will look as nice as that one… usually you will have to perform the above partial fraction procedure starting with **step one**, on the remainder!

2] What if the denominator doesn't factor into distinct binomials the way *you wish?* **Well, you have two options:**

First, you could scream and quit.

The advantage to this is that it will help you determine who else is feeling the same way you are.

I recommend the second; there are three basic ways that you will have a "bad" set of factors. Learn how to attack these three cases.

(Case A), One is if a factor in the denominator is linear and repeated,

$$(x-c)^n,$$

(Case B), the second is if a factor in the denominator is something like $(x^2 + 1)$, an irreducible quadratic *(it's called irreducible because it cannot be reduced, i.e., factored, further!)*

$$(x^2+bx+c),$$

a quadratic where the discriminant is negative, and

(Case C), worst of all, a factor in the denominator which is an irreducible quadratic raised to a power,

$$(x^2+bx+c)^n$$

(**Case A)** *SIMPLE EXAMPLE:* (Note the linear polynomial in the numerator and the repeated linear polynomial factor in the denominator, so denominator has higher degree than numerator, which means we don't have to do the process in 1] above!)

The fraction below can be split and re-written as shown so that it will be easier to integrate:

$$\frac{6x+7}{(x+3)^2}=\frac{A}{(x+3)}+\frac{B}{(x+3)^2}$$

Basically you just keep re-writing the denominator with increasing powers of the exponent until you reach the power of the original repeated linear term. In each additional term place another undetermined coefficient in the numerator. From here you can just apply an analogous technique to the above step by step method for finding the undetermined coefficients (A & B).

Starting at step 4.

$$6x + 7 = A(x + 3) + B$$

Choosing $x = -3$ yields B = -11 then choosing $x = 0 \Rightarrow 7 = 3A+-11$ or A=6.

If the original fraction had been $\frac{3x^2+5x-6}{(x+3)(x-2)^4}$ then the partial fraction form would be

$$\frac{3x^2+5x-6}{(x+3)(x-2)^4} = \frac{A}{(x+3)} + \frac{B_1}{(x-2)} + \frac{B_2}{(x-2)^2} + \frac{B_3}{(x-2)^3} + \frac{B_4}{(x-2)^4}$$

or

$$\frac{3x^2+5x-6}{(x+3)(x-2)^4} = \frac{A}{(x+3)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{E}{(x-2)^4}$$

Note that each of the above terms on the right is easy to integrate immediately. Of course the coefficients are still undetermined, and so that part is left to do too.

(Case B) *SIMPLE EXAMPLE:*
The fraction again can be split and re-written as shown so that it will be easier to integrate:

$$\frac{6x+7}{(x^2+4)(x+3)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x+3)}$$

The only really unique thing about this is that the factor with the quadratic in the denominator forces an arbitrary linear function $Ax + B$ in the corresponding term in the numerator.

Again, start back at step 4:

$$6x + 7 = (Ax + B)(x + 3) + C(x^2 + 4)$$

Choosing $x = -3 \Rightarrow -11=13C \Rightarrow C = -11/13$, so

$$6x + 7 = (Ax + B)(x + 3) + \frac{-11}{13}(x^2 + 4)$$

Then choosing $x = 0 \Rightarrow$ 7 = 3B – 4(11/13) $\Rightarrow$ B= 45/13

$$6x + 7 = (\text{A}x + \frac{45}{13})(x + 3) + \frac{-11}{13}(x^2 + 4)$$

Finally, choosing $x = -1 \Rightarrow$ 1= 2(-A+45/13)+(-11/13)(5) $\Rightarrow$ A=11/13

Then

$$\int \frac{6x+7}{(x^2+4)(x+3)}dx = \int \frac{\frac{11}{13}x + \frac{45}{13}}{(x^2+4)}dx + \int \frac{\frac{-11}{13}}{(x+3)}dx$$
$$= \frac{11}{13}\int \frac{x}{(x^2+4)}dx + \frac{45}{13}\int \frac{1}{(x^2+4)}dx + \frac{-11}{13}\int \frac{1}{(x+3)}dx$$

Clearly the middle integral is an inverse tangent! For the other two, we substitute

$$\begin{aligned} u_1 &= x^2 + 4 \\ du_1 &= 2x\,dx \end{aligned} \quad \text{and} \quad \begin{aligned} u_2 &= x + 3 \\ du_2 &= dx \end{aligned}$$

$$= \frac{1}{2}\left(\frac{11}{13}\right)\int \frac{1}{u_1}\,du_1 + \frac{1}{2}\left(\frac{45}{13}\right)\tan^{-1}\left(\frac{x}{2}\right) + \frac{-11}{13}\int \frac{1}{u_2}\,du_2$$

I am confident that you can finish the rest.

(Case C) NOT SO *SIMPLE EXAMPLE:*
The fraction again can be split and re-written as shown so that it will be easier to integrate:

$$\frac{1}{(x^3-1)(x^2+x+1)} = \frac{1}{(x-1)(x^2+x+1)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

This is similar to (case A)

$$1 = A(x^2+x+1)^2 + (Bx+C)(x^2+x+1)(x-1) + (Dx+E)(x-1)$$

Substituting in the consecutive values of x = -2, -1, 0, 1, 2 yields 5 equations

$$1 = 9A + 18B - 9C + 6D - 3E$$

$$1 = A + 2B - 2C + 2D - 2E$$

$$1 = A - C - E$$

$$1 = 9A$$

$$1 = 49A + 14B + 7C + 2D + E$$

These five equations must be solved which would take some patience. The answer is A = 1/9 , B = -1/9 , C = -2/9 , D= -1/3 , and E = -2/3 ,

Then

$$\frac{1}{(x^3-1)(x^2+x+1)} = \frac{\frac{1}{9}}{(x-1)} + \frac{\frac{-1}{9}x + \frac{-2}{9}}{x^2+x+1} + \frac{\frac{-1}{3}x + \frac{-2}{3}}{(x^2+x+1)^2}$$

Completing the square: $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

Let $u = x + \frac{1}{2}$ then $du = dx$ and $x = u - \frac{1}{2}$

So $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = u^2 + \frac{3}{4}$

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \int \frac{\frac{1}{9}}{(x-1)}dx + \int \frac{\frac{-1}{9}\left(u - \frac{1}{2}\right) + \frac{-2}{9}}{u^2 + \frac{3}{4}}dx + \int \frac{\frac{-1}{3}\left(u - \frac{1}{2}\right) + \frac{-2}{3}}{\left(u^2 + \frac{3}{4}\right)^2}dx$$

So

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\int \frac{1}{(x-1)}dx - \frac{1}{9}\int \frac{u}{u^2+\frac{3}{4}}du - \frac{1}{6}\int \frac{1}{u^2+\frac{3}{4}}du$$

$$-\frac{1}{3}\int \frac{u}{\left(u^2+\frac{3}{4}\right)^2}du - \frac{1}{2}\int \frac{1}{\left(u^2+\frac{3}{4}\right)^2}du$$

let $w = u^2 + \frac{3}{4} \;\Rightarrow\; dw = 2u\,du$

and let $v = x - 1 \Rightarrow\; dv = dx$ then

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\int \frac{1}{v}dv - \frac{1}{18}\int \frac{1}{w}dw - \frac{1}{6}\int \frac{1}{u^2+\frac{3}{4}}du$$

$$-\frac{1}{6}\int \frac{1}{w^2}dw - \frac{1}{2}\int \frac{1}{\left(u^2+\frac{3}{4}\right)^2}du$$

and integrating all but the last integral gives

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\ln|v| - \frac{1}{18}\ln|w| - \frac{1}{6}\left(\frac{1}{\sqrt{3/4}}\tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right)\right)$$

$$+\frac{1}{6}w^{-1} - \frac{1}{2}\int \frac{1}{\left(u^2+\frac{3}{4}\right)^2}du$$

Next, making a trig substitution of $u = \sqrt{3/4}\tan\theta$ for the final integral gives

$$\int \frac{1}{\left(u^2+\frac{3}{4}\right)^2}du = \frac{1}{(\sqrt{3/4})^3}\int \cos^2\theta d\theta = \frac{1}{(\sqrt{3/4})^3}\left[\frac{1}{2}(\cos\theta\sin\theta+\theta)\right]$$

$$= \frac{2}{3}\frac{u}{u^2+\frac{3}{4}} + \frac{1}{2(\sqrt{3/4})^3}\tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right)$$

Then FINALLY

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\ln|x-1| - \frac{1}{18}\ln\left|u^2+\frac{3}{4}\right| - \frac{1}{6}\left(\frac{1}{\sqrt{3/4}}\tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right)\right)$$

$$+\frac{1}{6}\left(u^2+\frac{3}{4}\right)^{-1} - \frac{1}{2}\left[\frac{2}{3}\frac{u}{u^2+\frac{3}{4}} + \frac{1}{2(\sqrt{3/4})^3}\tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right)\right] + C$$

but $u^2+\frac{3}{4} = x^2+x+1 \quad and \quad u = x+\frac{1}{2}$

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\ln|x-1| - \frac{1}{18}\ln|x^2+x+1| - \frac{1}{6}\left(\frac{1}{\sqrt{3/4}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3/4}}\right)\right)$$

$$+\frac{1}{6}(x^2+x+1)^{-1} - \frac{1}{2}\left[\frac{2}{3}\frac{x+1/2}{x^2+x+1} + \frac{1}{2(\sqrt{3/4})^3}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3/4}}\right)\right] + C$$

or combining like terms

$$\int \frac{1}{(x^3-1)(x^2+x+1)}dx = \frac{1}{9}\ln|x-1| - \frac{1}{18}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3/4}}\right)$$

$$-\frac{1}{3}\frac{x}{x^2+x+1} + C$$