

## Example 1: Find the 5<sup>th</sup> degree Taylor polynomial for $f(x) = \ln(x)$ expanded about 1.

STEP 1: Find out the degree  $n$  of the Taylor Polynomial that is required. This is  $n = 5$ .

STEP 2: Find and label  $f(x) = \ln(x)$  as the function that you are expanding and also the real number  $a = 1$  as the point that Taylor polynomial will be created about.

STEP 3: The basic form of the Taylor polynomial approximating  $f(x) = \ln(x)$  for  $x$  values close to  $a = 1$  is

$$\ln(x) \approx T_5(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3 + c_4(x-1)^4 + c_5(x-1)^5 \quad (1.1)$$

where  $c_0, c_1, c_2, \dots, c_n$  are constants that must be determined.

STEP 4: The “real work” is about to start. Get ready to take some derivatives, find and write out expressions

$$\begin{aligned} f'(x) &= \frac{1}{x}, \\ f''(x) &= \frac{-1}{x^2}, \\ f'''(x) &= \frac{2}{x^3}, \\ f^{(4)}(x) &= \frac{-6}{x^4}, \\ f^{(5)}(x) &= \frac{24}{x^5}, \\ f^{(6)}(x) &= \frac{-120}{x^6} \end{aligned} \quad (1.2)$$

STEP 5: Next, evaluate  $f(x), f'(x), f''(x), f'''(x), \dots, f^{(n)}(x)$  at the numerical value  $x = a = 1$  creating the numbers  $f(1), f'(1), f''(1), f'''(1), \dots, f^{(n)}(1)$ .

$$f(1) = \ln(1) = 0$$

$$f'(1) = 1,$$

$$f''(1) = -1,$$

$$f'''(1) = 2,$$

$$f^{(4)}(1) = -6,$$

$$f^{(5)}(1) = 24$$

STEP 6: Now we determine the constants  $c_0, c_1, c_2, \dots, c_n$  which we left as variables back in step 3, using the values calculated in step 5 by setting,

$$\begin{aligned}
 c_0 &= f(1) = \ln(1) = 0 \\
 c_1 &= f'(1) = 1, \\
 c_2 &= \frac{f''(1)}{2!} = \frac{-1}{2}, \\
 c_3 &= \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}, \\
 c_4 &= \frac{f^{(4)}(1)}{4!} = \frac{-6}{24} = \frac{-1}{4} \\
 c_5 &= \frac{f^{(5)}(1)}{5!} = \frac{24}{120} = \frac{1}{5},
 \end{aligned} \tag{1.3}$$

STEP 7: Substitute the numbers from step 6 in (1.3) into the expression (1.1), left in step 3, which is a 5<sup>th</sup> degree polynomial denoted  $T_5(x)$ , but don't expand it out just leave it in powers of  $x-1$ .

$$\ln(x) \approx T_5(x) = 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 \tag{1.4}$$

*If you are asked to determine the entire series, then you need to look for a pattern in the first few coefficients found in step 6 and then generalize it to all terms in the series.*

It is instructive to pursue this here. In this problem the  $c_k$  clearly have a sign change with odd + and even -, and also the denominator increases by 1 for each value of  $k$  so in general

$$\ln(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (x-1)^k,$$

but for what values of  $x$  will this series converge as  $k$  increases? Using the Basic Divergence Test (BDT) the series will *diverge* if the sequence defined by

$$a_k = (-1)^{k+1} \frac{1}{k} (x-1)^k \tag{1.5}$$

does not tend toward 0 as  $k$  approaches infinity. If

$$|x-1| > 1$$

using L'Hopital's Rule one can show that  $\lim_{k \rightarrow \infty} \frac{(x-1)^k}{k}$  will diverge to plus or minus

infinity, and thus series will diverge. If

$$|x-1| \leq 1 \Rightarrow x \in [0, 2]$$

one can show that  $\lim_{k \rightarrow \infty} \frac{(x-1)^k}{k}$  will converge to 0, and so the (BDT) is inconclusive.

However one can show for some of these values of  $x$  that the series *converges absolutely* using the Ratio Test. To see this

$$|a_k| = \frac{|x-1|^k}{k} \equiv b_k \quad (1.6)$$

again recall we are only interested in  $x \in [0, 2]$  as all the other values of  $x$  have been

ruled out. Using the Ratio Test on the series  $\sum_{k=1}^{\infty} b_k$  we examine

$$\frac{b_{k+1}}{b_k} = \frac{|x-1|^{k+1}}{\frac{k+1}{|x-1|^k}} = |x-1| \frac{k}{k+1} \rightarrow |x-1| \quad (1.7)$$

as  $k$  tends towards infinity. For any  $x \in (0, 2)$  the  $|x-1| < 1$  and so by the ratio test

$\sum_{k=1}^{\infty} b_k$  will converge which implies that  $\sum_{k=1}^{\infty} a_k$  converges absolutely. For  $x=0$  and  $x=2$

the test is inconclusive. Substituting  $x=0$  into  $a_k$  in (1.5),

$$a_k = (-1)^{k+1} \frac{1}{k} (0-1)^k = (-1)^{k+1} (-1)^k \frac{1}{k} = (-1)^{2k} (-1)^1 \frac{1}{k} = \frac{-1}{k}, \quad (1.8)$$

so that,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left( -\frac{1}{k} \right) = -\sum_{k=1}^{\infty} \frac{1}{k} \quad (1.9)$$

which diverges as a *p-series*. Substituting  $x=2$  into  $a_k$  in (1.5),

$$a_k = (-1)^{k+1} \frac{1}{k} (2-1)^k = (-1)^{k+1} (1)^k \frac{1}{k} = (-1)^{k+1} \frac{1}{k}, \quad (1.10)$$

so that,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \quad (1.11)$$

which can be shown to be an alternating series. By the Alternating Series Test this series will converge for as  $\frac{1}{k} \rightarrow 0$ , as  $k$  tends towards infinity. So we have been able to

deduce that the infinite series  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (x-1)^k$  actually converges to  $\ln(x)$  only when  $x$  is chosen in the  $(0, 2]$ .

STEP 8: If you are asked to *bound the error* when the polynomial,  $T_5$ , from step 7 is used to approximate  $f$  at some value  $x=b$ , suppose we randomly choose  $b=1.5$  then go back to step 4 and bring down your expression for  $f^{(6)}(x)$ . Find  $M$ , which is the largest

value that  $|f^{(6)}(x)|$  can take for any  $x$  in the interval between  $a = 1$  and  $b = 1.5$ . Of course that might take some effort, as essentially it is a maximization problem on a closed interval. The largest that

$$|f^{(6)}(x)| = \left| \frac{-120}{x^6} \right| = \left| \frac{120}{x^6} \right| \quad (1.12)$$

will be for  $x \in [1, 1.5]$  is when  $x = 1$  (“*Smaller denominator large fraction*”). So here we can choose  $M = \frac{120}{1^6} = 120$ , then the absolute error in using  $T_n(1.5)$  instead of  $f(1.5)$  is such that

$$|f(1.5) - T_n(1.5)| \leq \frac{120}{(5+1)!} |1.5 - 1|^{5+1} = \frac{120}{720} \cdot 5^6 = \frac{.5^6}{6} = 0.0026... \quad (1.13)$$

Now to see if this has actually worked:

$$f(1.5) = \ln(1.5) = 0.4054651081...$$

$$T_5(1.5) = 0.407291666.....$$

$$|f(1.5) - T_5(1.5)| = 0.0018265586....$$

so the error is actually less than bound of (1.13) which is a good thing!!