

Example 2: Find the 4th degree Taylor polynomial for $f(x) = \cos(\pi x)$ expanded about $\frac{1}{2}$.

STEP 1: Find out the degree n of the Taylor Polynomial that is required. This is $n = 4$.

STEP 2: Find and label $f(x) = \cos(\pi x)$ as the function that you are expanding and also the real number $a = \frac{1}{2}$ as the point that Taylor polynomial will be created about.

STEP 3: The basic form of the Taylor polynomial approximating $f(x) = \cos(\pi x)$ for x values close to $a = \frac{1}{2}$ is

$$\cos(\pi x) \approx T_4(x) = c_0 + c_1\left(x - \frac{1}{2}\right) + c_2\left(x - \frac{1}{2}\right)^2 + c_3\left(x - \frac{1}{2}\right)^3 + c_4\left(x - \frac{1}{2}\right)^4 \quad (1.1)$$

where $c_0, c_1, c_2, \dots, c_4$ are constants that must be determined.

STEP 4: The “real work” is about to start. Get ready to take some derivatives, find and write out expressions

$$\begin{aligned} f'(x) &= -\pi \sin(\pi x), \\ f''(x) &= -\pi^2 \cos(\pi x), \\ f'''(x) &= \pi^3 \sin(\pi x), \\ f^{(4)}(x) &= \pi^4 \cos(\pi x), \\ f^{(5)}(x) &= -\pi^5 \sin(\pi x) \end{aligned} \quad (1.2)$$

STEP 5: Next evaluate $f(x), f'(x), f''(x), f'''(x), \dots, f^{(n)}(x)$ at the numerical value

$x = a = \frac{1}{2}$ creating the numbers $f\left(\frac{1}{2}\right), f'\left(\frac{1}{2}\right), f''\left(\frac{1}{2}\right), f'''\left(\frac{1}{2}\right), \dots, f^{(n)}\left(\frac{1}{2}\right)$.

$$f\left(\frac{1}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'\left(\frac{1}{2}\right) = -\pi,$$

$$f''\left(\frac{1}{2}\right) = 0,$$

$$f'''\left(\frac{1}{2}\right) = \pi^3,$$

$$f^{(4)}\left(\frac{1}{2}\right) = 0$$

STEP 6: Now we determine the constants $c_0, c_1, c_2, \dots, c_n$ which we left as variables back in step 3, using the values calculated in step 5 by setting,

$$\begin{aligned}
 c_0 &= f\left(\frac{1}{2}\right) = \cos\left(\pi \frac{1}{2}\right) = 0 \\
 c_1 &= f'\left(\frac{1}{2}\right) = -\pi, \\
 c_2 &= \frac{f''\left(\frac{1}{2}\right)}{2!} = \frac{0}{2}, \\
 c_3 &= \frac{f'''\left(\frac{1}{2}\right)}{3!} = \frac{\pi^3}{6} = \frac{\pi^3}{6}, \\
 c_4 &= \frac{f^{(4)}\left(\frac{1}{2}\right)}{4!} = \frac{0}{24} = 0
 \end{aligned}
 \tag{1.3}$$

STEP 7: Substitute the numbers from step 6 in (1.3) into the expression (1.1), left in step 3, which is a 4th degree polynomial denoted $T_4(x)$, but don't expand it out just leave it in powers of $x - \frac{1}{2}$.

$$\begin{aligned}
 \cos(\pi x) \approx T_4(x) &= 0 + -\pi\left(x - \frac{1}{2}\right) + 0\left(x - \frac{1}{2}\right)^2 + \frac{\pi^3}{6}\left(x - \frac{1}{2}\right)^3 + 0\left(x - \frac{1}{2}\right)^4 \\
 &= -\pi\left(x - \frac{1}{2}\right) + \frac{\pi^3}{6}\left(x - \frac{1}{2}\right)^3
 \end{aligned}
 \tag{1.4}$$

STEP 8: If you are asked to *bound the error* when the polynomial, T_4 , from step 7 is used to approximate f at some value $x = b$, suppose we randomly choose $b = 3$ then go back to step 4 and write down your expression for $f^{(5)}(x)$. Find M , which is the largest value that $|f^{(5)}(x)|$ can take for any x in the interval between $a = \frac{1}{2}$ and $b = 3$. Of course, that might take some effort, as essentially it is a maximization problem on a closed interval. Notice that

$$|f^{(5)}(x)| = |-\pi^5 \sin(\pi x)| = \pi^5 |\sin(\pi x)| \leq \pi^5
 \tag{1.5}$$

since $-1 \leq \sin(\pi x) \leq 1$. So for $x \in [1/2, 3]$ we can choose $M = \pi^5$, and then the absolute error in using $T_4(3)$ instead of $f(3)$ is such that

$$|f(3) - T_4(3)| \leq \frac{\pi^5}{(4+1)!} \left|3 - \frac{1}{2}\right|^{4+1} = \frac{\pi^5}{120} (2.5)^5 = \frac{.5^6}{6} = 249.03... \quad (1.6)$$

Not great, so this suggests taking a higher degree polynomial other than degree 4. Now, to see if this has actually worked, .

$$f(3) = \cos(\pi 3) = -1.0$$

$$T_4(3) = 72.89.....$$

$$|f(3) - T_4(3)| = 73.89....$$

so the error is actually less than bound of (1.6) which is a good thing.

If you had the time to calculate $T_{30}(x)$ you would find that $T_{30}(3) = -1.00000064...$ which is a much better approximation.