TAYLOR SERIES WORKSHEET

RECALL: If f has infinitely many derivatives at c then the Taylor Series of f built about c has the form

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + \cdots$$

where we assume $|x-c|$ is within the radius of convergence . Notice first that

$f(c), f'(c), \frac{f''(c)}{2!}, \frac{f^{(3)}(c)}{3!}, \cdots,$ are just constants since all the functions are evaluated at c. If we stopped the series on the right hand side at the $(x-c)^n$ term in the sum, then typically $f(x)$ would just be *approximated* by the sum (which would be an n^{th} degree polynomial) and we would no longer have equality. If we let R represent the difference between this n^{th} degree polynomial and $f(x)$, (in other words " R is the remainder", or "the error", or the "crap left over"), then

$$f(x) = f(c) + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + R \qquad (1.1)$$

QUESTION: If you pick a different value of x would R typically change____________?

The GOAL of this worksheet is for you to show using results from calculus that for a given value of x, our remainder or error term R, takes the form

$$\boxed{R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}}$$

where ξ is between x and c.

QUESTION: Do you think this is going to be easy ____________?

First, some notation to make our lives a little easier: Let's let $P(x)$ represent everything on the right-hand-side (RHS) of (1.1) except the remainder term R. Then

$$f(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (1.2)$$

QUESTION: Since $f(c), f'(c), \frac{f''(c)}{2!}, \frac{f^{(3)}(c)}{3!}, \cdots,$ are just constants $P(x)$ is really just a polynomial. Imagine expanding out all the terms in $P(x)$, and gathering like terms what would the degree of this polynomial $P(x)$ be ?

____

So

$$P(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \qquad (1.3)$$

QUESTION: Recall *Rolle's theorem*. What does Rolle's theorem conclude if we have a function g that is continuous on $[a,b]$, differentiable on (a,b), and $g(a) = g(b) = 0$? Let me help: then there exists a point that we will call d in the open interval (a,b) such that

______________________.

Examine the *clearly and cleverly contrived function* g whose only independent variable is z, defined as

$$g(z) = f(z) - P(z) - R\frac{(z-c)^{n+1}}{(x-c)^{n+1}}. \tag{1.4}$$

If we replace $P(z)$ in (1.4) for what it really is using (1.3) we see that g take the form

$$g(z) = f(z) - \Big[\qquad\qquad\qquad\qquad\qquad\qquad \Big] - R\frac{(z-c)^{n+1}}{(x-c)^{n+1}} \tag{1.5}$$

MATH 235 ASIDE:

Let's take a momentary break from all this Taylor Series stuff and make a little trip back to Calculus I

QUESTION: If $h(y) = c_0 + c_1 y + c_2 y^2$ then $h''(y) =$ ____________ and $h'''(y) =$ _______

QUESTION: If $h(y) = c_0 + c_1 y + \cdots + c_4 y^4$ then $h^{(4)}(y) =$ ____________ and $h^{(5)}(y) =$ _______

QUESTION: If $h(y) = c_0 + c_1 y + \cdots + c_7 y^7$ then $h^{(7)}(y) =$ ____________ and $h^{(8)}(y) =$ _______

QUESTION: If $h(y) = c_0 + c_1 y + \cdots + c_n y^n$ then $h^{(n)}(y) =$ ____________ and $h^{(n+1)}(y) =$ _______

Now that we have had a little break let's get back to the task at hand.

QUESTION: What is the degree of the polynomial $P(z)$? ____________

QUESTION: So $P^{(n+1)}(z) =$ ____________?,

QUESTION: What would the $(n+1)^{st}$ derivative (with respect to z) of the function $(z-c)^{n+1}$ be equal to ?___________?

Now let's take the $(n+1)^{st}$ derivative of the function $g(z)$ and let's use the form in (1.4). Before you do this note that both R and $(x-c)^{n+1}$ DO NOT depend at all on the variable z. So as far as derivatives with respect to z are concerned they act like constants!

$$\begin{aligned} g^{(n+1)}(z) &= f^{(n+1)}(z) - P^{(n+1)}(z) - \underline{\hspace{6cm}}? \\ &= f^{(n+1)}(z) - \underline{\hspace{6cm}}? \end{aligned} \tag{1.6}$$

SUPPOSE that we knew ahead of time that there was a value of z that made $g^{(n+1)}(z)$ go to zero. For the sake of (our:) argument, let's just say that at $z=\xi$, $g^{(n+1)}(\xi)=0$. Then, using (1.6), we would be able to say that

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \underline{\hspace{6cm}}? \tag{1.7}$$

Now in (1.7), take $f^{(n+1)}(\xi)$ over to the LHS with the 0, and solve R. We would then have that

$$R = \frac{\hspace{3cm}}{\underline{\hspace{3cm}}} \quad ? \quad \text{YIPPEE☺} \tag{1.8}$$

We notice of course that this exactly what is in the box back on page 1. Are we done? NO!!!!!! Why not?? We *assumed* that there was a place where $g^{(n+1)}(z)$ evaluated to zero. And you know what happens when you *assume* something---that's right PROOF time!! To finish this worksheet, we need to justify that there actually is such a root. The remainder of the sheet attempts to show this.

What to do? What to do? What to do?

Go back to the equation that you wrote down for $g(z)$ in (1.5).

QUESTION: If you evaluate g at $z=c$ you find that $g(c) = \underline{\hspace{2.5cm}}$?

But if we evaluate g in (1.4) at $z=x$ we find that

$$g(x) = f(x) - P(x) - \underline{\hspace{1.5cm}}? \tag{1.9}$$

However back in (1.2) we have $f(x) = P(x) + R$, so

$$g(x) = \left(P(x)+R\right) - P(x) - \underline{\hspace{1.5cm}}? = \underline{\hspace{1.5cm}}? \tag{1.10}$$

Since f has at least $n+1$ derivatives then g will also have at least $n+1$ derivatives, and $g(c) = g(x) = 0$, so by Rolle's theorem there exists a point ξ_1 between x and c such that

$$g'\left(\xi_1\right) = \_\_\_\_\_\_\_? \tag{1.11}$$

So we know there is at least one point where $g'(z)$ is zero. If there where another point then we could actually use Rolle's theorem again. You guys must be worn out by now so let me take the derivative of g with respect to z using your expression for g in (1.5). Remember first however that $f(c), f'(c), \frac{f''(c)}{2!}, \frac{f^{(3)}(c)}{3!}, \cdots,$ are just constants, and using chain rule any derivative of the form $\left[\left(z-c\right)^k\right]' = k\left(z-c\right)^{k-1}$. After a little work we find

$$\begin{aligned} g'(z) = {} & f'(z) \\ & -\left[0 + f'(c) + f''(c)\left(z-c\right) + \frac{f'''(c)}{2!}\left(z-c\right)^2 + \cdots + \frac{f^{(n)}(c)}{\left(n-1\right)!}\left(z-c\right)^{n-1}\right] \\ & -\left(n+1\right)R\frac{(z-c)^n}{\left(x-c\right)^{n+1}} \end{aligned} \tag{1.12}$$

Now let's evaluate g' at $z=c$ to find that ,

$$g'(c) = f'(c) - \_\_\_\_\_\_\_? - \_\_\_\_\_\_? = \_\_\_\_\_\_? \tag{1.13}$$

Therefore in (1.11) and (1.13) we have that

$$g'(c) = g'(\xi_1) = \_\_\_\_\_\_? \tag{1.14}$$

So once again by ______________ theorem there must exist an ξ_2 between ξ_1 and c such that

$$g''(\xi_2) = \_\_\_\_\_\_\_\_? \tag{1.15}$$

I am the tired one now, so you find the second derivative of g with respect to z by differentiating both sides of (1.12) to find that

$$\begin{aligned} g''(z) = {} & f''(z) \\ & -\left[0 + f''(c) + \_\_\_\_\_\_?\left(z-c\right)^1 + \cdots + \_\_\_\_\_\_\_\_?\left(z-c\right)^{n-2}\right] \\ & - \_\_\_\_\_\_\_\_\_\_\_\_?\,R\frac{(z-c)^{n-1}}{\left(x-c\right)^{n+1}} \end{aligned} \tag{1.16}$$

Evaluating g'' at $z=c$ in (1.16) we see that

$$g''(c) = f''(c) - \_\_\_\_\_\_\_? - \_\_\_\_\_\_\_? = \_\_\_\_\_\_\_? \tag{1.17}$$

Therefore in (1.15) and (1.17) we have that

$$g''(c) = g''(\xi_2) = \underline{\hspace{2cm}}? \tag{1.18}$$

So once again by ______________ theorem there must exist an ξ_3 between ξ_2 and c such that

$$g'''(\xi_3) = \underline{\hspace{2cm}}? \tag{1.19}$$

We could iteratively keep on playing this game over and over again using Rolle's theorem for $n-2$ more times to show that had to be an ξ_n between ξ_{n-1} and ______ such that

$$g^{(n)}(\xi_n) = \underline{\hspace{2cm}}? \tag{1.20}$$

But differentiating (1.16) $n-2$ more times we would have that

$$\begin{aligned} g^{(n)}(z) = {} & f^{(n)}(z) \\ & - f^{(n)}(c) \\ & - \underline{\hspace{3cm}} R\frac{(z-c)}{(x-c)^{n+1}} \end{aligned} \tag{1.21}$$

and if we evaluate $g^{(n)}$ at $z = c$ using (1.21) we see that

$$g^{(n)}(c) = f^{(n)}(c) - \underline{\hspace{1.5cm}}? - \underline{\hspace{1.5cm}}? = \underline{\hspace{1.5cm}}? \tag{1.22}$$

Finally using (1.20) and (1.22) we have that

$$g^{(n)}(c) = g^{(n)}(\xi_n) = \underline{\hspace{2cm}}? \tag{1.23}$$

and by a final use of Rolle's theorem there must exist a point ξ between ξ_n and c such that

$$g^{(n+1)}(\xi) = 0. \tag{1.24}$$

Concluding, since ξ_1 is between x and c,
and ξ_2 is between ξ_1 and c,
and ξ_3 is between ξ_2 and c, $\cdots$
and ξ is between ξ_n and c, then ξ must also be between x and c.

WHEW!!!! Now, we are done, and we have proven that the remainder term in (1.1) has the form in the box! Another big YIPPEE is definitely in order ☺