The Product Rule for Differentiation: Suppose $f'(c)$ and $g'(c)$ exist and $h(x) = f(x)g(x)$ for all x in the domain of f and g, then show that $h'(c)$ exists and $h'(c) = f'(c)g(c) + f(c)g'(c)$.

$h'(c) = \lim_{x\to c} \frac{h(x)-h(c)}{x-c}$ by definition of derivative

$= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c}$ by evaluation of the given function $h(x) = f(x)g(x)$

$= \lim_{x\to c} \frac{f(x)g(x) + 0 - f(c)g(c)}{x-c}$ by algebra

$= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c}$ by algebra

$= \lim_{x\to c} \left(g(x)\frac{f(x)-f(c)}{x-c} + f(c)\frac{g(x)-g(c)}{x-c} \right)$ by algebra

$= \lim_{x\to c} \left(g(x)\frac{f(x)-f(c)}{x-c} \right) + \lim_{x\to c} \left(f(c)\frac{g(x)-g(c)}{x-c} \right)$ by *Sum Law* for **Limits**

$= \left(\lim_{x\to c} g(x)\right)\left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right) + \left(\lim_{x\to c} f(c)\right)\left(\lim_{x\to c} \frac{g(x)-g(c)}{x-c}\right)$ by *Product Law* for **Limits**

$= \left(\lim_{x\to c} g(x)\right) f'(c) + \left(\lim_{x\to c} f(c)\right) g'(c)$ by the *definition of derivative* and since, by our givens, $f'(c)$ and $g'(c)$ exist, and thus equal these limits.

$= \left(\lim_{x\to c} g(x)\right) f'(c) + f(c)g'(c)$ since the limit of the *constant* $f(c)$ is just the constant $f(c)$

$= g(c)f'(c) + f(c)g'(c)$ since $g'(c)$ exists (that is, g is differentiable at c) $\Rightarrow$ that we then *know* that g is continuous at c, and by definition of continuity, $\lim_{x\to c} g(x) = g(c)$.

Since $g(c), f'(c), f(c), g'(c)$ all exist (*meaning they are all real numbers*), then $h'(c)$ also exists, and so h is differentiable at c, and $h'(c)$ is computed from

$h'(c) = g(c)f'(c) + f(c)g'(c)$ OR $h'(c) = f(c)g'(c) + g(c)f'(c)$

Since c was *any arbitrary real number* at which f and g are differentiable, this is equivalent to:

$$\boxed{h(x) = f(x)g(x) \;\Rightarrow\; h'(x) = f(x)g'(x) + g(x)f'(x)}$$

(for any x s.t. $f'(x)$ and $g'(x)$ exist.)