Given:	**Prove:**
1) f is differentiable at $x=c$	h is differentiable at $x=c$
2) g is differentiable at $y=f(c)$	
3) $h(x)=g(f(x))$	

PROOF:

Since we were given in (1) that f is differentiable at $x=c$ we can use the definition of differentiability to state

$$\_\_\_\_? = \lim_{x\to c} \frac{\_\_\_\_? - f(c)}{x - \_\_\_?} \qquad (1.1)$$

To keep notation from getting overly cumbersome let's let

$$b = f(c) \qquad (1.2)$$

Then we can use given (2) that g is differentiable at $y=f(c)=b$ along with the definition of differentiability to state,

$$g'(b) = \lim_{y\to b} \frac{g(y) - \_\_\_\_?}{\_\_\_? - b}$$

Then using LT10,

$$\lim_{y\to b^-} \frac{g(y) - \_\_\_\_?}{\_\_\_? - b} = g'(\_\_\_\_?) \qquad (1.3)$$

and also

$$\lim_{y\to b^+} \frac{g(y) - \_\_\_\_?}{\_\_\_? - b} = g'(\_\_\_\_?). \qquad (1.4)$$

Next we define the quite strange looking function $P(y)$ by

$$P(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & if \quad y \neq b \\ \\ k \quad (\text{constant}) & if \quad y = b \end{cases} \tag{1.5}$$

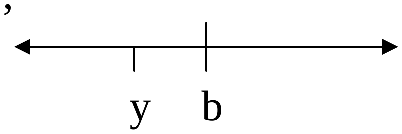
Let's see how well you understand piecewise defined functions,

$$P(b) = \_\_\_\_\_\_\_? \tag{1.6}$$

The rest of this page is devoted to the goal of finding a value for the constant k that will force P to be **continuous** at $y = b$. Let's see how well you understand continuity, we need

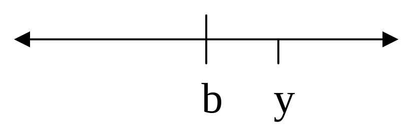
$$\begin{aligned} \lim_{y \to b} P(y) &= \_\_\_\_\_\_? \\ &= \_\_\_\_\_\_? \quad \text{by eqn(1.6)} \end{aligned} \tag{1.7}$$

So if we could find a value for $\lim_{y \to b} P(y)$ we could choose that value for k and our work for this page would be completed. However,



$$\begin{aligned} \lim_{y \to b^-} P(y) &= \lim_{y \to b^-} \frac{\_\_\_\_? - \_\_\_\_?}{\_\_\_\_? - \_\_\_\_?} \quad \text{by def(n) of } P \\ &= \_\_\_\_\_\_? \quad \text{by eqn(1.3)} \end{aligned}$$

and



$$\begin{aligned} \lim_{y \to b^+} P(y) &= \lim_{y \to b^+} \frac{\_\_\_\_? - \_\_\_\_?}{\_\_\_\_? - \_\_\_\_?} \quad \text{by def(n) of } P \\ &= \_\_\_\_\_\_? \quad \text{by eqn(1.4)} \end{aligned}$$

Using LT10, then $\lim_{y \to b} P(y) = \_\_\_\_\_?$ So choosing

$$k = \_\_\_\_\_\_? \tag{1.8}$$

forces P to be **continuous** at $y = b$.

With k now chosen we next evaluate P from (1.5) at $y = f(x)$ and find that

$$P\big(f(x)\big) = \begin{cases} \dfrac{g(\_\_\_\_?) - g(b)}{\_\_\_\_? - b} & if \quad f(x) \neq b \\ \\ g'(b) & if \quad f(x) = b \end{cases} \tag{1.9}$$

CASE1:

Now suppose $x \neq c$ and $\boxed{f(x) \neq f(c)}$, which also implies $f(x) \neq b$ since we set $b = f(c)$ earlier, then

$$\frac{g\big(f(x)\big) - g\big(f(c)\big)}{x - c} = \frac{g\big(f(x)\big) - g\big(f(c)\big)}{f(x) - f(c)} \bullet \frac{\_\_\_\_? - \_\_\_\_?}{x - c}$$

$$= \frac{g\big(f(x)\big) - g\big(b\big)}{f(x) - b} \bullet \frac{\_\_\_\_? - \_\_\_\_?}{x - c}$$

← Just fill in the same answers from the line above

$$= P\big(\_\_\_\_?\big) \bullet \frac{\_\_\_\_? - \_\_\_\_?}{x - c}$$

CASE2:

Now suppose $x \neq c$ but $\boxed{f(x) = f(c)}$, which again also implies $f(x) = b$, then

$$\frac{g\big(f(x)\big) - g\big(f(c)\big)}{x - c} = \frac{g\big(\_\_\_\_?\big) - g\big(f(c)\big)}{x - c} = \frac{0}{x - c} = 0$$

However also then,

$$P\big(f(x)\big) \bullet \frac{f(x) - f(c)}{x - c} = P\big(\_\_\_\_?\big) \bullet \frac{\_\_\_\_? - f(c)}{x - c}$$

$$= P\big(b\big) \bullet \frac{0}{x - c}$$

$$= \_\_\_\_? \bullet 0 = 0$$

← You do **NOT** just want to fill in $P(b)$ again use eqn(1.9).

So in this second case we see that also

$$\frac{g\big(f(x)\big) - g\big(f(c)\big)}{x - c} = P\big(f(x)\big) \bullet \frac{f(x) - f(c)}{x - c}$$

since both are equal to 0.

Case 1 and Case 2 combined show that if $x \neq c$ we have for our very clever function P that

$$\frac{g\left(f(x)\right)-g\left(f(c)\right)}{x-c}=P\left(f(x)\right)\bullet\frac{f(x)-f(c)}{x-c} \qquad (1.10)$$

Now we are ready to make the argument!

$h'(c)=\lim\limits_{x\to c}\dfrac{h(x)-h(c)}{x-c}$ by def(n) of ______________

$=\lim\limits_{x\to c}\dfrac{g\left(f(x)\right)-g\left(f(c)\right)}{x-c}$ by given ______and________

$=\lim\limits_{x\to c}P\left(f(x)\right)\bullet\dfrac{f(x)-f(c)}{x-c}$ by eqn ______and LT9

$=\lim\limits_{x\to c}P\left(f(x)\right)\bullet\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ by LT ______

$=\lim\limits_{x\to c}P\left(f(x)\right)\bullet$______? by eqn1.1

$=P\left(\lim\limits_{x\to c}f(x)\right)\bullet$______? by _______________ as_____ is cont. by p.2

$=P\left(\text{_____?}\right)\bullet$______? as diff. of f at c implies cont. of f at c

$=P\left(\text{_____?}\right)\bullet$______? by def(n) of b

$=$______? $\bullet f'(c)$ by eqn1.5 and eqn1.8

$=$______ $\bullet$______? by def(n) of b

This shows that h is differentiable at $x=c$ since given 1says $f'(c)$ and given 2 says $g'\left(f(c)\right)$ exists. Thus, in general, from our givens:

$$\text{IF} \quad h(x)=g\left(f(x)\right)$$
$$\text{THEN} \quad h'(x)=g'\left(f(x)\right)\bullet f'(x) \qquad (1.11)$$

CHAIN-RULE:
for differentiating composition of functions.

$$\left[g\left(f(x)\right)\right]'=g'\left(f(x)\right)\bullet f'(x)$$