5.

Given: 1) $\lim_{x\to c} f(x) = L$
2) $\lim_{x\to c} f(x) = M$

Prove: $L = M$

PROOF:

(By contradiction) Assume $L \neq M$ and try to show that this leads to a contradiction of something known to be true. Since we assume $L \neq M$ then either $L < M$ or $M < L$ and either one of these would imply $|L-M| > 0$ and $\frac{|L-M|}{2} > 0$.

Since $\frac{|L-M|}{2} > 0$ and we were given $\lim_{x\to c} f(x) = L$ there exists **by the definition of limit** $\delta_1 > 0$ such that

$$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{|L-M|}{2}, \qquad (1.1)$$

and as we were given $\lim_{x\to c} g(x) = M$ there also exists **by the definition of limit** $\delta_2 > 0$ such that

$$0 < |x-c| < \delta_2 \Rightarrow |f(x) - M| < \frac{|L-M|}{2} \qquad (1.2)$$

Setting $\delta = \min(\delta_1, \delta_2)$ then by definition of *min*

$$\delta \le \delta_1 \quad \& \quad \delta \le \delta_2 \quad ,$$

and using(1.1) and (1.2) we have also

$$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \frac{|L-M|}{2} \qquad (1.3)$$

and

$$0 < |x-c| < \delta \Rightarrow |f(x) - M| < \frac{|L-M|}{2}. \qquad (1.4)$$

Now using (1.3) and (1.4) for any x in the set $0 < |x-c| < \delta$

$$\begin{aligned}
|L-M| &= |L+0-M| \\
&= |L - f(x) + f(x) - M| \\
&\le |L - f(x)| + |f(x) - M| && \text{as } |a+b| \le |a| + |b| \\
&\le |f(x) - L| + |f(x) - M| && \text{as } |a-b| = |b-a| \\
&< \frac{|L-M|}{2} + \frac{|L-M|}{2} && \text{using } (1.3)\,\&\,(1.4) \\
&= |L-M|
\end{aligned}$$

Carefully following the equalities and inequalities backwards we see that the last statement just showed that

$$|L-M|<|L-M|$$

which clearly is not true and represents a contradiction. This means that a statement we assumed to be true (that was not part of the givens) is false. However the only such assumption was that $L \neq M$ thus $L = M$. $\square$

6.

Given: 1) $\lim_{x\to c} f(x) = L$ 2) $L \neq 0$ 3) $\lim_{x\to c} g(x) = 0$	**Prove:** $\lim_{x\to c} \frac{f(x)}{g(x)}$ *D.N.E.*

PROOF:

(By contradiction) Assume $\lim_{x\to c} \frac{f(x)}{g(x)}$ DOES EXIST and show that leads to a contradiction. If $\lim_{x\to c} \frac{f(x)}{g(x)}$ DOES EXIST then $\lim_{x\to c} \frac{f(x)}{g(x)} = M$ where $M \in \mathbb{R}$. Then since $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and the $\lim_{x\to c} g(x)$ exists

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}g(x)\right) = \lim_{x\to c}\frac{f(x)}{g(x)}\lim_{x\to c} g(x) = M\cdot 0 = 0 .$$

BUT it is also true that

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}g(x)\right) = \lim_{x\to c} f(x) = L$$

using the last two equations we see

$$L = \lim_{x\to c}\left(\frac{f(x)}{g(x)}g(x)\right) = 0 .$$

which implies $L = 0$ which contradicts $L \neq 0$ (given2). This means that a statement we assumed to be true (that was not part of the givens) is false. However the only such assumption was that $\lim_{x\to c} \frac{f(x)}{g(x)}$ DOES EXIST thus $\lim_{x\to c} \frac{f(x)}{g(x)}$ DOES NOT EXIST.

$\square$

7.

Given:	Prove:
1) f is continuous at c 2) g is continuous at $f(c)$ 3) $h(x) = g(f(x))$	h is continuous at c

PROOF:

Since g is continuous at $f(c)$ **by definition of continuity**

$$\lim_{y \to f(c)} g(y) = g(f(c)),$$

so if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|y - f(c)| < \delta_1 \;\Rightarrow |g(y) - g(f(c))| < \varepsilon, \tag{1.5}$$

or stating (1.5) with interval notation,

$$y \in \left(f(c) - \delta_1, f(c) + \delta_1\right) \;\Rightarrow g(y) \in \left(g\left(f(c)\right) - \varepsilon, g\left(f(c)\right) + \varepsilon\right). \tag{1.6}$$

Since f is continuous at c then **by definition of continuity**

$$\lim_{x \to c} f(x) = f(c),$$

and for the $\delta_1 > 0$ in (1.5) there exists $\delta > 0$ such that

$$\begin{aligned} |x - c| < \delta \;&\Rightarrow |f(x) - f(c)| < \delta_1 \\ &\Rightarrow f(x) \in \left(f(c) - \delta_1, f(c) + \delta_1\right) \end{aligned}. \tag{1.7}$$

and using (1.6) substituting $f(x)$ for y we see that,

$$\begin{aligned} &\Rightarrow g\left(f(x)\right) \in \left(g\left(f(c)\right) - \varepsilon, g\left(f(c)\right) + \varepsilon\right) \\ &\Rightarrow \left|g\left(f(x)\right) - g(f(c))\right| < \varepsilon \\ &\Rightarrow |h(x) - h(c)| < \varepsilon \end{aligned} \tag{1.8}$$

where h was defined by the third given. Following the implies symbols through (1.7) and (1.8) we have that

$$|x - c| < \delta \;\Rightarrow |h(x) - h(c)| < \varepsilon. \tag{1.9}$$

which is equivalent to showing

$$\lim_{x \to c} h(x) = h(c). \tag{1.10}$$

Proving that h is continuous at c.

□