

p -Coloring Classes of Torus Knots

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Abstract

We develop a theorem for determining the p -colorability of any (m, n) torus knot. We also prove that any p -colorable (m, n) torus knot has exactly one p -coloring class. Finally, we show that every p -coloring of the braid projection of an (m, n) torus knot must use all of the p colors.

MR Subject Classifications: 57M27, 05C15

Keywords: torus knot, colorability, braid word, variance

1 Introduction

Our first result is a theorem specifically determining the p -colorability of any (m, n) torus knot. It has been previously shown that a $(m, m - 1)$ torus knot is always p -colorable for p equal to m or $m - 1$ depending on which is odd (see [5] and [10]). Another proven result is that a $(2, n)$ torus knot is always p -colorable for p equal to n and a $(3, n)$ torus knot is always 3-colorable if n is even [10]. A result similar to ours was also stated as a lemma without proof in [3].

Our second result shows that any p -colorable (m, n) torus knot has only one p -coloring class. A general result investigating colorings of torus knots by finite Alexander quandles appears in [2]. Our result is a special instance of this result; however, we present a proof using only elementary techniques. p -coloring classes have also previously been investigated in relationship to pretzel knots by [4].

Using our second result, we were also able to show a minor result concerning the distribution of colors in a p -coloring of a torus knot. We showed that any p -coloring of the braid representation of an (m, n) torus knot must use each of the p colors. Distribution of colors in p -colorings of knots has been

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previously investigated with the Kauffman-Harary Conjecture. This conjecture is concerned with the distribution of colors in a p -coloring of an alternating knot with prime determinant. Asaeda, Przytcki, and Sikora prove the Harary-Kauffman Conjecture is true for pretzel knots and Montesinos knots in [1].

2 Notation

Let $G_p(K)$ be the set of all p -colorings for a knot K . Note that $G_p(K)$ is empty if K is not p -colorable. We wish to count the number of p -colorings in $G_p(K)$ that differ by more than just a permutation of the colors. To do this precisely, suppose \mathcal{S}_K is the set of all strands of K . A p -coloring of a knot K is a map $\gamma : \mathcal{S}_K \rightarrow \mathbb{Z}_p$ satisfying the condition that $2\gamma(s_j) - \gamma(s_i) - \gamma(s_k) = 0 \pmod{p}$ for all $s_i, s_j, s_k \in \mathcal{S}_k$ at a crossing of K , where s_j is the overcrossing strand and s_i, s_k are the undercrossing strands. It is an easy exercise to see that the relation \sim defined by $\gamma \sim \delta \iff \gamma = \rho \circ \delta$ for some permutation $\rho : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ for $\gamma, \delta \in G_p(K)$ is an equivalence relation on $G_p(K)$.

The p -coloring class of $\gamma \in G_p(K)$ is the set $\bar{\gamma} = \{\delta \in G_p \mid \delta \sim \gamma\}$. Two p -colorings are said to be *equivalent* if they are in the same p -coloring class, and *fundamentally different* if they are in different p -coloring classes. The set of p -coloring classes for a given knot K will be denoted by $C_p(K)$. The number of p -coloring classes for K will be denoted by $|C_p(K)|$.

Let $T_{m,n}$ represent the torus knot characterized by the number of times m that it circles around the meridian of the torus and the number of times n that it circles around the longitude of the torus. A torus knot $T_{m,n}$ is a knot and not a link if and only if m and n are relatively prime. The torus knot $T_{m,n}$ can be drawn as the closure of the n -strand braid word $(\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1})^m$. We will refer to the word $(\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1})$ as the *base* for the braid word of $T_{m,n}$, and say that a *cycle* is a single completion of the base for the braid word of a knot $T_{m,n}$. For the duration of the paper, any braid representation of a knot $T_{m,n}$ is considered to have n strands and m cycles, where the 0^{th} strand is the strand on the far left of the braid representation.

3 p -Colorability of Torus Knots

The following theorem completely characterizes the p -colorability of torus knots.

Theorem 1. *Suppose $T_{m,n}$ is a torus knot and p is prime.*

- i) If m and n are both odd, then $T_{m,n}$ is not p -colorable.*
- ii) If m is odd and n is even, then $T_{m,n}$ is p -colorable if and only if $p \mid m$.*
- iii) If m is even and n is odd, then $T_{m,n}$ is p -colorable if and only if $p \mid n$.*

Note that if $T_{m,n}$ is a torus knot, as opposed to a link, then m and n are relatively prime, and thus m and n cannot both be even. Results similar to those in Theorem 1 were stated without proof by Asami and Satoh in [3].

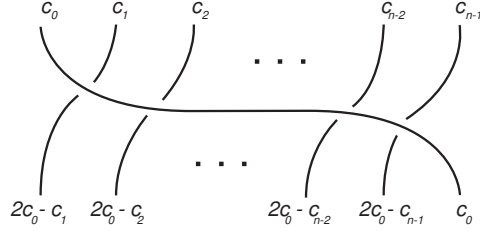


Figure 1: The action of ϕ on the j^{th} color array of $T_{m,n}$.

To prove Theorem 2 it suffices to show that if $T_{m,n}$ is p -colorable, then every p -coloring of $T_{m,n}$ is equivalent. To this end, in Section 4.1 we will exhibit a specific p -coloring of $T_{m,n}$ called the *main p -coloring*. Then in Section 4.2 we will prove Theorem 2 by showing that all p -colorings of $T_{m,n}$ are equivalent to the main p -coloring.

4.1 A main p -coloring for every torus knot

Given a torus knot $T_{m,n}$, consider two p -colorings $\alpha, \beta \in G_p(T_{m,n})$. Let α_i^j and β_i^j be the colors of the i^{th} strand of any braid knot K after j cycles for the p -colorings α and β , respectively. Note that the p -colorings α and β are in the same coloring class in $C(T_{m,n})$ if and only if

$$\alpha_i^j = \alpha_l^k \iff \beta_i^j = \beta_l^k \quad (3)$$

for all $i, l \in \{0, 1, \dots, n-1\}$ and $j, k \in \{0, 1, \dots, m\}$.

Given a p -colored braid representation of a knot $T_{m,n}$, we will say that the j^{th} color array of $T_{m,n}$ is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color of the i^{th} strand of the braid representation of $T_{m,n}$ after j cycles. For a knot $T_{m,n}$ to be p -colorable it is a necessary and sufficient condition that the initial color array of its braid representation be exactly the same as its final color array.

Let $(c_0, c_1, \dots, c_{n-1})$ be the j^{th} color array for the braid representation of some knot $T_{m,n}$, and consider the map $\phi : (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}_p)^n$ defined by

$$\phi(c_0, c_1, \dots, c_{n-1}) = (2c_0 - c_1, 2c_0 - c_2, \dots, 2c_0 - c_{n-1}, c_0). \quad (4)$$

Notice that ϕ is the map that, given the j^{th} color array of a knot $T_{m,n}$, returns the $(j+1)^{\text{st}}$ color array according the rules of p -colorability, as seen in Figure 1. Define the map ϕ^j to be the composition of j copies of ϕ , or in other words:

$$\phi^j(\text{initial color array of } T_{m,n}) = j^{\text{th}} \text{ color array of } T_{m,n}.$$

A p -coloring of any knot $T_{m,n}$ is entirely determined by its initial color array in a braid representation for $T_{m,n}$. Furthermore, for a knot $T_{m,n}$ to be p -colorable, it is a necessary and sufficient condition that we have $\phi^m = \text{id}$ when applied to the initial color array of the braid representation.

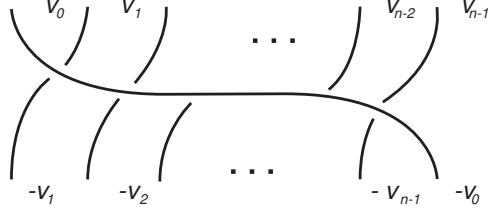


Figure 2: The action of ψ on the j^{th} variance vector of $T_{m,n}$.

We now consider a second n -tuple that can be defined from the braid representation of a p -colored knot $T_{m,n}$. The *color variance* between any two adjacent strands in the projection colored with c_i and c_j respectively is $c_j - c_i \pmod p$. (We consider the far left and far right strands to be adjacent.) Given a p -colored braid representation of a knot $T_{m,n}$, the j^{th} *variance vector* of $T_{m,n}$ is the element of $(\mathbb{Z}_p)^n$ whose i^{th} component is the color variance between the $(i-1)^{\text{th}}$ and i^{th} strands after j cycles. The 0^{th} variance vector of $T_{m,n}$ is referred to as the *initial variance vector*. A *constant variance vector* is a variance vector $V = (v_0, v_1, \dots, v_{n-1})$ where $v_0 = v_1 = \dots = v_{n-1}$. If $(c_0, c_1, \dots, c_{n-2}, c_{n-1})$ is the j^{th} color array for some knot $T_{m,n}$, then the j^{th} variance vector for $T_{m,n}$ is

$$(v_0, v_1, \dots, v_{n-2}, v_{n-1}) = (c_1 - c_0, c_2 - c_1, \dots, c_{n-1} - c_{n-2}, c_0 - c_{n-1}). \quad (5)$$

Let $\psi: (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}_p)^n$ denote the map that takes as input the j^{th} variance vector of a knot $T_{m,n}$ and returns the $(j+1)^{\text{st}}$ variance vector. By Equations (4) and (5) we have:

$$\begin{aligned} \psi(v_0, v_1, \dots, v_{n-2}, v_{n-1}) &= ((2c_0 - c_2) - (2c_0 - c_1), (2c_0 - c_3) - (2c_0 - c_2), \\ &\quad \dots, c_0 - (2c_0 - c_{n-1}), (2c_0 - c_1) - c_0) \\ &= (c_1 - c_2, c_2 - c_3, \dots, c_{n-1} - c_0, c_0 - c_1) \\ &= (-v_1, -v_2, \dots, -v_{n-1}, -v_0). \end{aligned}$$

Figure 2 shows how Equation (6) looks as we move from the j^{th} to the $(j+1)^{\text{th}}$ variance vector. In this figure, the color variance v_i between the $(i-1)^{\text{st}}$ and i^{th} strands is shown between those two strands.

We define $\psi^j: (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}_p)^n$ to be the composition of j copies of ψ . Since for every application of ψ to $(v_0, v_1, \dots, v_{n-1})$, all entries get multiplied by -1 and move over one position to the left while wrapping around, we have that $\psi^j(v_0, v_1, \dots, v_{n-1})$ acts such that for all $i \in \{0, 1, \dots, n-1\}$

$$v_i \longrightarrow (-1)^j v_{i+j \pmod n}. \quad (6)$$

In a more general form this means that

$$\psi^j(v_0, v_1, \dots, v_{n-1}) = \begin{cases} (v_j, v_{j+1}, \dots, v_0, v_1, \dots, v_{j-1}), & \text{if } j \text{ is even} \\ (-v_j, -v_{j+1}, \dots, -v_0, -v_1, \dots, -v_{j-1}), & \text{if } j \text{ is odd,} \end{cases}$$

where all subscripts are taken modulo n .

Given a p -colorable torus knot $T_{m,n}$, we can assume without loss of generality that m is even and n is odd. We will now exhibit a specific p -coloring of $T_{m,n}$ for every prime p that divides n , called the *main p -coloring* of $T_{m,n}$. Since p divides n there exists $r \in \mathbb{Z}^+$ such that $n = rp$. Consider the braid representation of $T_{m,n}$ with n strands and m cycles whose initial, or 0th, color array M is the n -tuple whose entries are the series $0, 1, \dots, p-2, p-1$, repeated r times:

$$M = (0, 1, \dots, p-2, p-1, 0, 1, \dots, p-1, \dots, p-2, p-1). \quad (7)$$

Using Equation (4) we can deduce that the 1st color array of $T_{m,n}$ is

$$\begin{aligned} \phi(M) &= \phi(0, 1, \dots, p-2, p-1, 0, 1, \dots, p-1, \dots, p-2, p-1) \\ &= (2(0) - 1, 2(0) - 2, \dots, 2(0) - (p-1), 2(0) - 0, \\ &\quad 2(0) - 1, 2(0) - 2, \dots, 2(0) - 0, \dots, 2(0) - (p-1), 0) \\ &= (-1, -2, \dots, -(p-1), 0, -1, -2, \dots, 0, \dots, -(p-1), 0) \\ &= (p-1, p-2, \dots, 1, 0, p-1, p-2, \dots, 0, \dots, 1, 0). \end{aligned}$$

Note that the 2nd color array of $T_{m,n}$ is

$$\begin{aligned} \phi^2(M) &= \phi(\phi(M)) \\ &= \phi(p-1, p-2, \dots, 1, 0, p-1, p-2, \dots, 0, \dots, 1, 0) \\ &= (2(p-1) - (p-2), 2(p-1) - (p-3), \dots, 2(p-1) - 0, \\ &\quad 2(p-1) - (p-1), 2(p-1) - (p-2), 2(p-1) - (p-3), \dots, \\ &\quad 2(p-1) - (p-1), \dots, 2(p-1) - 0, (p-1)) \\ &= (p, p+1, \dots, 2p-2, 2p-1, p, p+1, \dots, 2p-1, \dots, 2p-2, p-1) \\ &= (0, 1, \dots, p-2, p-1, 0, 1, \dots, p-1, \dots, p-2, p-1) \\ &= M. \end{aligned}$$

4.2 Every p -coloring of a torus knot is equivalent to the main p -coloring

To prove Theorem 2 it suffices to show that if $T_{m,n}$ is p -colorable, then every p -coloring of $T_{m,n}$ is equivalent to the main p -coloring of $T_{m,n}$. Notice that the initial variance vector for the main p -coloring of a knot $T_{m,n}$ as defined in Equation (7) is the constant variance vector $(1, 1, \dots, 1)$. Our first step in proving Theorem 2 will be to prove the following lemma.

Lemma 2. *Suppose $T_{m,n}$ is a p -colorable torus knot where p is prime, and consider the n -strand braid representation of $T_{m,n}$. If the initial color array of the p -coloring has a constant variance vector, then that initial color array induces a p -coloring that is equivalent to the main p -coloring of $T_{m,n}$.*

Proof. Let C be an initial color array of $T_{m,n}$ with constant variance vector (v, v, \dots, v) . The for some $a \in \mathbb{Z}_p$ we can write C as

$$C = (a + 0v, a + v, \dots, a + (n-2)v, a + (n-1)v). \quad (8)$$

The first color array of $T_{m,n}$ with this p -coloring is

$$\begin{aligned} \phi(C) &= \phi(a, a + v, \dots, a + (n-2)v, a + (n-1)v) \\ &= (2a - (a + v), 2a - (a + 2v), \dots, 2a - (a + (n-1)v), a) \\ &= (a - v, a - 2v, \dots, a - (n-1)v, a) \\ &= (a + (n-1)v, a + (n-2)v, \dots, a + v, a). \end{aligned} \quad (9)$$

Notice that ϕ reverses the order of the entries in C . Applying ϕ again, we see that the second color array is

$$\begin{aligned} \phi^2(C) &= \phi(a - v, a - 2v, \dots, a - (n-1)v, a) \\ &= (2(a - v) - (a - 2v), 2(a - v) - (a - 3v), \dots, 2(a - v) - a, a - v) \\ &= (a, a + v, \dots, a + (n-3)v, a - 2v, a - v) \text{notag} \end{aligned} \quad (10)$$

$$\begin{aligned} &= (a, a + v, \dots, a + (n-3)v, a + (n-2)v, a + (n-1)v) \\ &= C. \end{aligned} \quad (11)$$

Thus we can see that $\phi^2 = \text{id}$ when applied to color arrays with constant variance.

Since p is prime, there exists $t \in \mathbb{Z}_p$ such that $a = tv \pmod{p}$. Thus, $C = (tv, tv+v, tv+2v, \dots, tv+(n-1)v) = (tv, (t+1)v, (t+2)v, \dots, (t+(n-1))v)$. We know if $v \in \mathbb{Z}_p/\{0\}$ and p is prime, then $\langle v \rangle = \mathbb{Z}_p$. Therefore the first p entries of C are distinct elements of \mathbb{Z}_p . Moreover, since p divides n , C is comprised of this exact pattern of all of \mathbb{Z}_p repeated precisely $\frac{n}{p}$ times. Therefore,

$$C = (c_0, c_1, \dots, c_{p-2}, c_{p-1}, c_0, c_1, \dots, c_{p-1}, \dots, c_{p-2}, c_{p-1}) \quad (12)$$

where c_0, c_1, \dots, c_{p-1} repeats $\frac{n}{p}$ times. Since c_0, c_1, \dots, c_{p-1} are distinct, we know that for $i \in \{0, 1, \dots, p-1\}$ we have

$$c_i = c_j \iff i = j. \quad (13)$$

Using the notation in Equation (13), and the results in Equations (9) and (11), we see that the j^{th} color array for the p -coloring induced by the initial color array C is

$$\phi^j(C) = \begin{cases} (c_0, c_1, \dots, c_{p-2}, c_{p-1}, c_0, c_1, \dots, c_{p-1}, \dots, c_{p-2}, c_{p-1}), & \text{if } j \text{ is even} \\ (c_{p-1}, c_{p-2}, \dots, c_1, c_0, c_{p-1}, c_{p-2}, \dots, c_0, \dots, c_1, c_0), & \text{if } j \text{ is odd.} \end{cases}$$

On the other hand, from Section 4.1 we know that the j^{th} color array for the main p -coloring induced by the initial color array M from Equation (7) is

$$\phi^j(M) = \begin{cases} (0, 1, \dots, p-2, p-1, 0, 1, \dots, p-1, \dots, p-2, p-1), & \text{if } j \text{ is even} \\ (p-1, p-2, \dots, 1, 0, p-1, p-2, \dots, 0, \dots, 1, 0), & \text{if } j \text{ is odd.} \end{cases}$$

We wish to prove that C and M induce equivalent p -colorings in terms of the condition in Equation (3). Let $\pi_i : (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}_p)$ be the i^{th} projection map, and let $\phi_i^j = \pi_i \circ \phi^j : (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}_p)$. Notice that ϕ_i^j takes in an initial color array and returns the color of the i^{th} strand of the j^{th} color array of the induced p -coloring. From the expression for $\phi^j(C)$ above, we see that for $j \in \{0, 1, \dots, m\}$ and $i \in \{0, 1, \dots, n-1\}$ we have

$$\phi_i^j(C) = \begin{cases} c_i \bmod p, & \text{if } j \text{ is even} \\ c_{p-1-i} \bmod p, & \text{if } j \text{ is odd.} \end{cases} \quad (14)$$

Similarly, from the expression for $\phi^j(M)$ above, we see that

$$\phi_i^j(M) = \begin{cases} i \bmod p, & \text{if } j \text{ is even} \\ p-1-i \bmod p, & \text{if } j \text{ is odd.} \end{cases} \quad (15)$$

It is now easy to see from Equations (13), (14), and (15) that we have

$$\phi_i^j(M) = \phi_l^k(M) \iff \phi_i^j(C) = \phi_l^k(C)$$

for $j, k \in \{0, 1, \dots, m\}$ and $i, l \in \{0, 1, \dots, n-1\}$. Therefore by Equation (3) we know that C and M induce equivalent p -colorings on $T_{m,n}$. \square

We will now use Lemma 2 to prove Theorem 2. The key will be to show that there cannot be a p -coloring of $T_{m,n}$ that does not have a constant variance vector.

Proof. Suppose $T_{m,n}$ is a p -colorable torus knot, and consider the n -strand braid representation of $T_{m,n}$. By Theorem 1 we can assume without loss of generality that n is odd and m is even, and that p divides n . Seeking a contradiction, assume that $|C_p(T_{m,n})| > 1$. Specifically, assume there is a p -coloring $\gamma \in G_p(T_{m,n})$ that is fundamentally different from the main p -coloring of $T_{m,n}$. Let $G = (g_0, g_1, \dots, g_{n-2}, g_{n-1})$ be the initial color array for γ , and let $V = (v_0, v_1, \dots, v_{n-2}, v_{n-1}) = (g_1 - g_0, g_2 - g_1, \dots, g_{n-1} - g_{n-2}, g_0 - g_{n-1})$ be the initial variance vector for γ .

Using ψ as defined in Equation (6) we can apply ψ^q to V to get the q^{th} variance vector of γ (where all subscripts are taken mod n):

$$\psi^q(V) = \begin{cases} (v_q, v_{q+1}, \dots, v_0, v_1, \dots, v_{q-1}), & \text{if } q \text{ is even} \\ (-v_q, -v_{q+1}, \dots, -v_0, -v_1, \dots, -v_{q-1}), & \text{if } q \text{ is odd.} \end{cases}$$

Let r be the smallest positive integer for which V partitions into s repeating sections of length r . Note that $1 \leq r \leq n$ and that $n = rs$. Since by hypothesis γ is not equivalent to the main p -coloring of $T_{m,n}$, Lemma 2 tells us that the variance vector V is not constant. Therefore we have $r > 1$.

Since m is even, we know from Equation (4) that

$$\phi^m(V) = (v_m, v_{m+1}, \dots, v_0, v_1, \dots, v_{m-1}).$$

In other words, ϕ^m turns the initial variance vector V into the vector where all entries have been shifted to the left m positions and wrapped around. Because m is the number of cycles in our braid representation, and γ is a p -coloring of $T_{m,n}$, we must have $\psi^m(V) = V$. In order for this to occur, ψ^m must shift V over by some multiple of r , the length of a repeating section. Therefore we must have $m = kr$ for some $k \in \mathbb{Z}^+$.

We have now shown that $n = rs$, $m = kr$, and $r > 1$; these facts imply that $\gcd(m, n) \geq r > 1$. But this contradicts our assumption that $T_{m,n}$ is a knot and not a link. Therefore, there cannot exist a p -coloring γ that is fundamentally different from the main coloring, and hence we must have $|C_p(T_{m,n})| = 1$. \square

Because the main p -coloring of a torus knot $T_{m,n}$ uses all p colors, and every p -coloring that is equivalent to the main coloring must also use all p colors, we have the following immediate corollary to Theorem 2.

Corollary 3. *Every p -coloring of a braid projection of knot $T_{m,n}$ must use all p colors.*

5 Questions for future research

1. What is the significance of the color distribution in Corollary 3 and what other knots have a similar color distribution?
2. Do variance vectors have useful applications to other types of knots, especially those whose braid word is some power of a base word?
3. What other types of p -colorable knots have only one p -coloring class?

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