CHEEGER CONSTANTS OF PLATONIC GRAPHS

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Abstract. The Platonic graphs π_n arise in several contexts, most simply as a quotient of certain Cayley graphs associated to the projective special linear groups. We show that when $n = p$ is prime, π_n can be viewed as a complete multigraph in which each vertex is itself a wheel on $n + 1$ vertices. We prove a similar structure theorem for the case of an arbitrary prime power. These theorems are then used to obtain new upper bounds on the Cheeger constants of these graphs. These results lead immediatley to similar results for Cayley graphs of the group $PSL(2, \mathbb{Z}_n)$.

1. Introduction

Let G be the group $\mathbb{Z}_n \times \mathbb{Z}_n - \{(0, 0)\}$. We define the graphs π_n as follows: The vertex set of π_n is given by $G/\{\pm 1\}$. Two vertices (a, b) and (c, d) are connected by an edge if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 \pmod{n}.$$

The graph π_n is called the n-th Platonic graph and is easily seen to be n-regular. Platonic graphs are interesting for many reasons, their strong expansion properties being most prominent [2]. Their name derives from the fact that when $n = 3, 4, 5$ the graphs π_n correspond to the 1-skeletons of the Platonic solids composed of triangles, squares or pentagons respectively; namely the tetrahedron, the cube, and the dodecahedron.

An alternative characterization of these graphs arises from a consideration of certain quotients of the projective special linear groups. Set $\Gamma(N) = PSL(2, \mathbb{Z}_n)$. If we set

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then the set $\Omega = \{U, U^{-1}, V\}$ generates $\Gamma(N)$ and we can construct the Cayley graph of Γ with respect to these generators. We will denote this graph by G_n, and we note that it is three-regular. We can define an equivalence relation on $V(G_n)$ by declaring $v_1 \sim v_2$ if and only if $v_1 = v_2 U^k$ for some positive integer k. The equivalence classes of this relation are given by circuits of length n. The graphs π_n are obtained from G_n by collapsing these circuits to a point. Still other ways of characterizing these graphs can be found in [2].

Let G be a finite, connected, multigraph with vertex set $V(G)$ and edge set $E(G)$. If $S \subset V(G)$, then we define the boundary of S, denoted by ∂S, to be the subset of $E(G)$ containing precisely those edges with one endpoint in S and the other endpoint in the complement of S. The isoperimetric number of G, denoted by $i(G)$, is then defined by

$$i(G) = \inf_{S} \frac{|\partial S|}{|S|},$$

where the infimum is taken over all sets S satisfying $|S| \leq \frac{1}{2}|V(G)|$. The fraction $\frac{|\partial S|}{|S|}$ is called the isoperimetric quotient of S. The isoperimetric number finds numerous applications in combinatorics; for example, it can be used to derive bounds on the eigenvalues of G (see [7] and [9], for example). In general it is difficult to compute $i(G)$ explicitly, and one must make do with bounding it in terms of other graph properties.

In [3], Brooks established the following bounds:

Theorem 1.1. *Let p be a prime satisfying $p \equiv 1 \pmod 4$. Then*

$$\frac{p^2 - 2p + 5}{4(p-1)} \leq i(\pi_n) \leq \frac{(p-1)p}{2(p+1)}.$$

Note that the upper bound approaches $\frac{p}{2}$, and the lower bound approaches $\frac{p}{4}$, as $p \to \infty$.

In the present paper we prove several theorems regarding the structure of these graphs for arbitrary prime powers, and use these theorems to derive upper bounds on $i(\pi_{p^r})$.

2. The Main Results

Let C_{n-1} be a cycle of length $n-1$, and let W_n denote the graph obtained from C_{n-1} by the addition of a single vertex v, with one edge connecting v to each of the $n-1$ vertices in C_{n-1}. The graphs W_n are sometimes referred to as wheel graphs (this terminology and notation is taken from [1]). Let K_n denote the complete graph on n vertices, and let K_n^m denote the complete multi-graph on n vertices, with m edges connecting each pair of distinct vertices.

We prove the following:

Theorem 2.1. *When p is prime, π_p can be partitioned into $\frac{p-1}{2}$ isomorphic copies of W_{p+1}, with $2p$ edges joining every pair of wheels. Alternatively, π_p is the complete multigraph $K^{2p}_{\frac{p-1}{2}}$, in which each vertex should be viewed as a wheel.*

Theorem 2.2. *Let p^r be an arbitrary prime power and let ϕ denote Euler's phi-function. Then the graph π_{p^r} can be partitioned into two sets A, B so that A is isomorphic to the complete multigraph $K^{2p^r}_{\frac{\phi(p^r)}{2}}$ as described in Theorem 2.1, while vertices in B are joined only to vertices in A.*

Note that since the vertices in π_{p^r} correspond to disjoint cycles in the Cayley graph of $PSL(2,\mathbb{Z}_{p^r})$, Theorem 2.2 leads immediately to a structure theorem for these Cayley graphs as well. We can also use our structure theorems can to find upper bounds on the isoperimetric numbers $i(\pi_{p^r})$ of these graphs.

Theorem 2.3.

$$i(\pi_{p^r}) \leq \begin{cases} \frac{p^r(p-1)}{2(p+1)} & \text{if } p \not\equiv 3 \pmod 4, \\ \frac{p^{2r}-2p^{2r-1}+5p^{2r-2}-4p^{r-1}+4}{2(p^r-2p^{r-1}-3p^{r-2}+4p^{-1})} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1}$$

We note that both of these bounds approach $\dfrac{p^r}{2}$ from below as $p \to \infty$.

3. Graph Theoretic Preliminaries

If v_1, v_2 are adjacent vertices of G we will write $v_1 \sim v_2$. Let K be a finite group and let Ω be a generating set for K. If we have $\Omega = \Omega^{-1}$ we say Ω is symmetric. Then the Cayley graph of K with respect to the symmetric generating set Ω, denoted by $G(K, \Omega)$, is the graph with vertex set K, with two vertices k_1, k_2 joined by an edge if we have $k_1 = k_2\omega$ for some $\omega \in \Omega$. Cayley graphs are vertex-transitive and $|K|$-regular. The set $\Omega = \{U, U^{-1}, V\}$ defined in the previous section is a symmetric generating set for $PSL(2, \mathbb{Z}_n)$. More precisely,

$$PSL(2, \mathbb{Z}_n) \cong \langle u, v | u^n = v^2 = (uv)^3 = 1 \rangle.$$

We have $|PSL(2, \mathbb{Z}_n)| = \dfrac{n^3}{2} \prod_{p|n} \left(1 - \dfrac{1}{p^2}\right)$. A proof of this can be found in [8].

Given graphs G, H, we construct a new graph G' as follows: Let $\{H_v\}_{v \in V(G)}$ be a set of $|G|$ many isomorphic copies of H, indexed by the vertices of G. Put

$$V(G') = \bigsqcup_{v \in V(G)} V(H_v).$$

Now for every pair $(v_1, v_2) \in E(G)$, we choose arbitrary vertices w_1, w_2, subject to the restrictions $w_1 \in H_{v_1}$ and $w_2 \in H_{v_2}$, and put $(w_1, w_2) \in E(G')$. The set of all graphs G' constructable in this way will be denoted by $\Phi_H(G)$.

Some elements of $\Phi_H(G)$ are subgraphs of the more familiar Cartesian product of G and H; i.e. the graph with vertex set $V(G) \times V(H)$ with $(g, h) \sim (g', h')$ if $g \sim g'$ and $h = h'$, or

$g = g'$ and $h \sim h'$. They can be pictured by imagining that you are given the graph G, but when you zoom in closely on the vertices of G you find that each is actually a copy of H.

It was shown in [3] that

$$\langle u\rangle\backslash PSL(2,\mathbb{Z}_n) \cong \langle -1\rangle\backslash\left\{\begin{pmatrix}a & b\end{pmatrix} \neq \begin{pmatrix}0 & 0\end{pmatrix} \mid a,b\in\mathbb{Z}_n \quad ((a,b),n)=1\right\}.$$

In other words, the set of n-cycles generated from multiplying an arbitrary vertex by powers of u can be indexed by the bottom rows of matrices in $PSL(2,\mathbb{Z}_n)$. The following lemma was also shown in [3]:

Lemma 1. *The vertex* $\begin{pmatrix}a & b\end{pmatrix}$ *is adjacent to* $\begin{pmatrix}c & d\end{pmatrix}$ *if and only if* $\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} \equiv \pm 1 \pmod{n}$.

Finally, let G'_n be the graph obtained from G_n by contracting the n-cycles of G_n to a single vertex. Clearly, $G_n \in \Phi_{C_n}(G'_n)$. Note that G' is regular of degree n and $|G'_n| = |G_n|/n$. It is easily seen that $G'_n \cong \pi_n$, the Platonic graphs defined earlier.

4. The Structure of π_{p^r}

In this section we will establish some results regarding the structure of π_n, when n is an arbitrary prime power. These results will be essential for producing our candidate isoperimetric sets.

Let $a \in (Z_n)^\times, 0 \le a \le [n/2]$, where $[\cdot]$ is the greatest integer function. Denote by $H(a)$ the subgraph of π_n induced by the following subset of $V(G')$:

$$V(H(a)) = \left\{\begin{pmatrix}0 & a\end{pmatrix}, \begin{pmatrix}a^{-1} & b\end{pmatrix} \mid b \in (\mathbb{Z}_n)^\times\right\}.$$

We have $|H(a)| = (n+1)$. It is clear that if a and a' both satisfy the requirements of the above definition, then $H(a) \cong H(a')$.

Lemma 2. $H(a) \cong W_{n+1}$.

Proof. For any $b \in \mathbb{Z}_n$, we know from Lemma 1 that the vertex $\begin{pmatrix} 0 & a \end{pmatrix}$ is adjacent to $\begin{pmatrix} a^{-1} & b \end{pmatrix}$, which accounts for n edges. Since $\deg \begin{pmatrix} 0 & a \end{pmatrix} = n$ we know it is adjacent to no other vertices.

If $\begin{pmatrix} a^{-1} & b \end{pmatrix}$ is adjacent to $\begin{pmatrix} a^{-1} & x \end{pmatrix} \in H(a)$ then $x = \pm a + b$. Therefore, we have a cycle of adjacent vertices

$$\begin{pmatrix} a^{-1} & b \end{pmatrix}, \begin{pmatrix} a^{-1} & a+b \end{pmatrix}, \ldots, \begin{pmatrix} a^{-1} & (n-1)a+b \end{pmatrix}, \begin{pmatrix} a^{-1} & b \end{pmatrix},$$

and this exhausts all the vertices and edges of $H(a)$. □

Now let H be the subgraph of π_n induced by $\bigcup_a H(a)$, where $a \in (Z_n)^\times$ and $0 < a \le [n/2]$. Let ϕ denote Euler's phi-function.

Proposition 1. $H \in \Phi_{W_{n+1}}\left(K^{2n}_{\frac{\phi(n)}{2}}\right)$.

Proof. First note that $V(H) = \bigcup_a V(H(a))$. Since there are $\dfrac{\phi(n)}{2}$ possible values of a, we conclude that H contains that many isomorphic copies of $H(a) \cong W_{n+1}$.

Now consider a, $c \in (\mathbb{Z}_n)^\times$ where $a \ne c$ and $0 < a, c \le [n/2]$. We set

$$v_1 = \begin{pmatrix} c^{-1} & a^{-1} + ca^{-1}b \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} c^{-1} & -a^{-1} + ca^{-1}b \end{pmatrix}.$$

We note that $v_1, v_2 \in H(c)$ and $v_1 \ne v_2$. Then for $\begin{pmatrix} a & b \end{pmatrix} \in H(a)$ we have

$$\det \begin{pmatrix} a & b \\ c & \pm a^{-1} + ca^{-1}b \end{pmatrix} \equiv \pm 1 \pmod{n}.$$

It follows that $\begin{pmatrix} a & b \end{pmatrix} \in H(a)$ is adjacent to vertices v_1 and v_2 in $H(c)$, and it is easy to see that these are the only vertices in $H(c)$ adjacent to $\begin{pmatrix} a & b \end{pmatrix}$. We conclude that any vertex of the form $\begin{pmatrix} a & b \end{pmatrix}$ is adjacent to exactly two vertices in $H(c)$. Since there are n such vertices in $H(a)$, we conclude that for any given choice of c, $H(a)$ has $2n$ edges incident with vertices in $H(c)$.

Finally, we observe that $a, c \in (\mathbb{Z}_n)^\times$ were arbitrary. Thus, any two wheels in H are joined by precisely $2n$ edges. It follows that if each wheel were contracted to a point, we would be left with a graph isomorphic to $K^{2n}_{\frac{\phi(n)}{2}}$. Therefore, $H \in \Phi_{W_{n+1}}\left(K^{2n}_{\frac{\phi(n)}{2}}\right)$ and the proof is complete. □

Proposition 2. $|H| = |\pi_n|$ *if and only if* $n = p$.

Proof. We have $|H| = |\pi_n|$ if and only if

$$|H| = \frac{(n+1)}{2}\left(n\prod_{p|n} 1 - \frac{1}{p}\right) = \frac{n^2}{2}\prod_{p|n} 1 - \frac{1}{p^2} = |\pi_n|.$$

This equation is true if and only if

$$n\prod_{p|n}\frac{p+1}{p} = n+1.$$

Since this last equation is true if and only if $n = p$ the proof is complete. □

The following corollary is an immediate consequence of Proposition 2. Recall that G_n is the Cayley graph of $PSL(2, \mathbb{Z}_n)$ defined earlier.

Corollary 1. *For any prime* p *we have* $G_p \in \Phi_{C_p}\left(\Phi_{W_{p+1}}(K^{2p}_{\frac{p-1}{2}})\right)$.

Recognizing π_p as being essentialy a complete multigraph leads easily to the following bounds on the isoperimetric number $i(\pi_p)$:

Proposition 3. *Let* p *be a prime. Then*

$$i(\pi_p) \leq \frac{2p\left[\frac{p+1}{4}\right]}{p+1}.$$

Proof. There are $\frac{p-1}{2}$ copies of W_{p+1} in π_p. Let S contain any $\left[\frac{p-1}{4}\right]$ of them. Then it is a simple calculation to show that

$$|\partial S| = 2p \left[\frac{p-1}{4}\right] \left[\frac{p+1}{4}\right]$$
$$|S| = (p+1) \left[\frac{p-1}{4}\right]$$

from which the result follows immediately. □

When $p \equiv 1 \pmod 4$, this result recovers the bound from [3].

Things are more complicated when we move to the case where $n = p^r$ for some $r > 1$. We can still construct the graph H as before, but now we will have vertices not contained in $H(a)$ for any a. However, the next lemma makes a strong statement regarding the arrangement of those vertices.

Lemma 3. *Let r be an integer with $r > 1$. Then no vertices in $\pi_{p^r} - H$ are adjacent.*

Proof. We begin by determining a method for indexing the vertices of $\pi_{p^r} - H$. Specifically, we show that

$$\pi_{p^r} - H = \langle -1 \rangle \backslash \left\{ \begin{pmatrix} pa & b \end{pmatrix} \mid a \in \mathbb{Z}_{p^{r-1}} - \{0\}, \quad b \in (\mathbb{Z}_{p^r}^{\times}) \right\}. \tag{1}$$

It follows from our previous work that

$$|H| = \frac{\phi(p^r)}{2}(p^r + 1) = \frac{p^r(p^r+1)}{2}\left(1 - \frac{1}{p}\right).$$

Counting vertices on the LHS of (1) gives,

$$|\pi_{p^r} - H| = \frac{p^{2r}}{2}\left(1 - \frac{1}{p^2}\right) - \frac{p^r(p^r+1)}{2}\left(1 - \frac{1}{p}\right).$$

Thus, we have that

$$|\pi_{p^r} - H| = \frac{p^{r-1}}{2}\left(p^r - p^{r-1} - p + 1\right).$$

On the RHS of (1) we find there are $\dfrac{\phi(p^r)}{2}$ possible values for $b \in (\mathbb{Z}_{p^r})^\times$, where $\pm b$ are viewed as equivalent. Also, we have $|\mathbb{Z}_{p^{r-1}} - \{0\}| = p^{r-1} - 1$ possible values of a. The RHS therefore contains

$$\frac{\phi(p^r)}{2}\left(p^{r-1}-1\right) = \frac{p^r}{2}\left(1-\frac{1}{p}\right)\left(p^{r-1}-1\right)$$

vertices. This last expression is equal to $|\pi_{p^r} - H|$. Since every vertex of the form $\begin{pmatrix} pa & b \end{pmatrix}$ with a and b in the appropriate range is in $\pi_{p^r} - H$, we are done.

To prove the lemma we need only observe now that

$$\det\begin{pmatrix} pa & b \\ pa' & b' \end{pmatrix} \equiv 0 \pmod{p}.$$

It follows that the vertices $\begin{pmatrix} pa & b \end{pmatrix}$ and $\begin{pmatrix} pa' & b' \end{pmatrix}$ are not adjacent. □

5. Constructing an Isoperimetric Set

5.1. Primes Not Congruent to Three mod Four. We first assume $p \not\equiv 3 \pmod 4$. In this case $\phi(p^r)/2$ is even and $\left(\dfrac{-1}{p}\right) = 1$, where $\left(\dfrac{\cdot}{p}\right)$ is the Legendre symbol.

Lemma 4. *Let $p \not\equiv 3 \pmod 4$. If $\begin{pmatrix} pa & b \end{pmatrix} \in \pi_{p^r} - H$ is adjacent to a vertex in $H(b')$ then $\left(\dfrac{b}{p}\right) = \left(\dfrac{b'}{p}\right)$.*

Proof. Since the Legendre symbol is multiplicative, and since $b \in (\mathbb{Z}_{p^r})^\times$, we see that $\left(\dfrac{b}{p}\right) = \left(\dfrac{b^{-1}}{p}\right)$ and similarly for b'. Now if $\det\begin{pmatrix} pa & b \\ b' & x \end{pmatrix} \equiv \pm 1 \pmod{p^r}$ then $pax - bb' \equiv \pm 1 \pmod{p^r}$. Further, $pax - bb' \equiv 1 \pmod{p}$. It follows that $b' \equiv \pm b^{-1} \pmod{p}$. Since $\left(\dfrac{-1}{p}\right) = 1$ the proof is complete. □

We can now construct our candidate isoperimetric set. Let

$$(\pi_{p^r} - H)_\pm = \left\{ \begin{pmatrix} pa & b \end{pmatrix} \in \pi_{p^r} - H \;\middle|\; \left(\frac{b}{p}\right) = \pm 1 \right\}.$$

Note that

$$|(\pi_{p^r} - H)_+| = |(\pi_{p^r} - H)_-| = \frac{1}{2}|\pi_{p^r} - H| = \frac{\phi(p^r)}{4}(p^{r-1} - 1).$$

Now let

$$H_{\pm} = \left\{ H(b) \middle| b \in (\mathbb{Z}_{p^r}^{\times}), 0 \leq b \leq [p^r/2], \left(\frac{b}{p}\right) = \pm 1 \right\}.$$

We observe that

$$|H_+| = |H_-| = \frac{1}{2}|H| = \frac{\phi(p^r)}{4}(p^r + 1).$$

It follows from Lemma 4 that vertices in $(\pi_{p^r} - H)_+$ are adjacent only to vertices in H_+ and vertices in $(\pi_{p^r} - H)_-$ are adjacent only to vertices in H_-. We also note that the four sets $(\pi_{p^r} - H)_{\pm}, H_{\pm}$ partition $V(\pi_n)$.

We set $S = H_+ \bigcup (\pi_{p^r} - H)_+$ as our candidate isoperimetric set. It is now necessary to compute $|S|$ and $|\partial S|$. For the former observe:

$$|S| = |(\pi_{p^r} - H)_+| + |H_+| = \frac{\phi(p^r)}{4}\left(p^r + p^{r-1}\right)$$

To compute the latter we first note that since the vertices in $(\pi_{p^r} - H)_+$ are only adjacent to vertices in H_+, they contribute nothing to ∂S. Recall that for any appropriate b, b', each of p^r vertices in $H(b)$ sends two edges to $H(b')$. Since we have $\phi(p^r)/4$ choices for both b and b', it follows that

$$|\partial S| = 2p^r \left(\frac{\phi(p^r)}{4}\right)^2 = \frac{p^r\left[\phi(p^r)\right]^2}{8}.$$

Therefore,

$$i(\pi_{p^r}) \leq \frac{|\partial S|}{|S|} = \frac{p^r \frac{[\phi(p^r)]^2}{8}}{\frac{\phi(p^r)}{4}\left(p^r + p^{r-1}\right)} = \frac{p^r \phi(p^r)}{2p^{r-1}(p+1)} = \frac{p^r(p-1)}{2(p+1)}.$$

Note that when $r = 1$ this bound matches the result obtained by Brooks, Perry, and Petersen in [3].

5.2. **Primes Congruent to Three mod Four.** The situation is more difficult when $p \equiv 3 \pmod 4$, as now the Legendre symbol is not invariant with respect to multiplication by minus one. This makes it far more difficult to define a good bipartition of the vertices of π_{p^r}. However, our starting point is the same as in the previous sections. We still have our multigraph H as before, with outlying vertices in $\pi_{p^r} - H$ adjacent only to vertices in H.

Let

$$T = \left\{ b \in (\mathbb{Z}_p)^\times / \pm \{1\} \middle| 1 \leq b \leq \frac{p-1}{2} \right\}.$$

In what follows we will not make any distinction between a particular integer and the two element coset it represents. Thus, b and $-b$ will be considered equivalent. Then for any $t \in T$ we can find a unique $t' \in T$ such that $tt' \equiv \pm 1 \pmod p$. It is easily seen that $t = t'$ implies $t = 1$. Therefore, we can partition $T - \{1\}$ into two sets, which we will denote by T_+ and T_-, so that if $tt' \equiv \pm 1 \pmod p$ then t, t' are in different sets.

Set $\mathcal{G} = (\mathbb{Z}_{p^r})^\times / \pm \{1\}$. We observe that any $b \in \mathcal{G}$ has a base p expansion; we write $b = a + \sum_{i=1}^{r-1} b_i p^i$, where $0 \leq a, b_i \leq p-1$. Alternatively, $b = a + cp$ for some constant c. Also, we note that either $a = 1$ or $a \in T - \{1\}$. We use these facts to partition $\mathcal{G}$ as follows: Define

$$B_\pm = \{ b = a + cp \in \mathcal{G} \mid a \in T_\pm \}.$$

Also define

$$B_1 = \left\{ 1 + \sum_{i=1}^{r-1} b_i p^i \mid 0 \leq b_i \leq p-1 \right\}.$$

Then

$$\mathcal{G} = B_+ \bigsqcup B_- \bigsqcup B_1.$$

As before, we index elements of $\pi_{p^r} - H$ by matrices with bottom row $\begin{pmatrix} pa & b \end{pmatrix}$, with $b \in (\mathbb{Z}_{p^r})^\times$. Vertices in the wheel having center $\begin{pmatrix} 0 & b^{-1} \end{pmatrix}$ are indexed by matrices having bottom row $\begin{pmatrix} b & c \end{pmatrix}$. This induces partitions of $\pi_{p^r} - H$ and H, each into three pieces, according to whether $b \in B_+, B_-, B_1$. The resulting six sets will be denoted by $(\pi_{p^r} - H)_{+,-,1}$ and $H_{+,-,1}$.

Lemma 5. *If the vertex* $\begin{pmatrix} pa & b' \end{pmatrix} \in (\pi_{p^r} - H)$ *is adjacent to the vertex* $\begin{pmatrix} b & c \end{pmatrix} \in H(b^{-1})$ *then exactly one of the following is true:*

(1) *One of* b, b' *is in* B_+ *and the other is in* B_-.

(2) *Both* b *and* b' *are in* B_1.

Proof. The vertices $\begin{pmatrix} pa & b' \end{pmatrix}$ and $\begin{pmatrix} b & c \end{pmatrix}$ are adjacent precisely when $pac - bb' \equiv \pm 1 \pmod{p^r}$. As before, this implies that $bb' \equiv 1 \pmod{p}$. Suppose that $b \in B_1$. Then $b = 1 + px$ for some constant x. If we write $b' = a + x'p$, then $bb' = a + py$, for some constant y. This is congruent to ± 1 modulo p only if $a = 1$, in which case $b' \in B_1$ as well. The remainder of the lemma now follows from our definitions of B_+ and B_-. □

This lemma establishes that vertices in $(\pi_{p^r} - H)_{+,-,1}$ are adjacent only to vertices in $H_{-,+,1}$ respectively. If S is our candidate isoperimetric set of π_{p^r}, then we will place one of $(\pi_{p^r} - H)_+ \bigcup H_-$ and $(\pi_{p^r} - H)_- \bigcup H_+$ in S and the other in the complement of S. The boundary will then contain all of the edges joining H_+ to H_-. It remains to find an appropriate bipartition of H_1 and $(\pi_{p^r} - H)_1$.

The centers of wheels in H_1 are indexed by matrices of the form $\begin{pmatrix} 0 & b \end{pmatrix}$, where $b \in (\mathbb{Z}_{p^r})^\times$ and $b \equiv 1 \pmod{p}$. There are precisely p^{r-1} such numbers, which implies there are an odd number of wheels in H_1. Since the wheels in H are arranged in the form of a complete multigraph, we will arbitrarily assign $\frac{p^{r-1}-1}{2}$ of them to S, and the remainder to the complement of S. Denote these sets by H_{1+} and H_{1-} respectively.

We now consider the vertices in $(\pi_{p^r} - H)_1$.

Lemma 6. *Let S be an isoperimetric set for π_{p^r}. Let $\mathcal{C}_S$ denote the number of elements in ∂S incident with a vertex in $(\pi_{p^r} - H)_1$. Then*

$$|\mathcal{C}_S| \leq \frac{p^{3r-2} - p^{2r-1}}{2}.$$

Proof. It is a simple computation to show that $|(\pi_{p^r} - H)_1| = p^{2r-2} - p^{r-1}$. This implies there are $p^{3r-2} - p^{2r-1}$ edges incident with vertices in $(\pi_{p^r} - H)_1$. All of these edges are also incident with vertices in H_1. Since $|H_{1+}| \leq \frac{1}{2}|H_1|$, we conclude that $\mathcal{C}$ contains no more than half of these edges, and the proof is complete. □

We now observe that $|H|$ is on the order of p^{2r}, while $|(\pi_{p^r} - H)_1|$ is on the order of p^{2r-2}. As a consequence, the size of $(\pi_{p^r} - H)_1 \bigcap S$ will not significantly affect our estimate of the isoperimetric number. So it will not be necessary to partition $(\pi_{p^r} - H)_1$.

Therefore, we now construct the following set:

$$S = (\pi_{p^r} - H)_- \bigcup H_+ \bigcup H_{1+}.$$

To evaluate its isoperimetric quotient we must determine $|S|$ and $|\partial S|$. Note: Given a subset A of vertices in S, we will denote by $\partial_A(S)$ the set of edges in ∂S incident with vertices in A. Further note that two vertices are considered identical if they differ by multiplication by ± 1.

Our previous work allows us to compute:

$$|(\pi_{p^r} - H)_-| = \frac{1}{2}\left(p^{r-1} - 1\right)\left(\frac{\phi(p^r)}{2} - p^{r-1}\right)$$

$$|H_+ \bigcup H_{1+}| = (p^r + 1)\left[\frac{\phi(p^r) - 2}{4} + \frac{p^{r-1} + 1}{2}\right].$$

It follows that

$$|S| = \frac{1}{4}\left(p^{2r} - 2p^{2r-1} - 3p^{2r-2} + 4p^{r-1}\right).$$

Computing $|\partial S|$ requires the following:

Lemma 7. *If S is an isoperimetric set for the complete multigraph K_y^x, where y is odd, then*

$$|\partial S| = x\left(\binom{\frac{y-1}{2}}{}(y-1) - 2\binom{\frac{y-1}{2}}{2}\right),$$

where $\binom{\cdot}{\cdot}$ denotes the binomial coefficient.

Proof. An isoperimetric set for a complete graph on y vertices has size $\frac{1}{2}(y-1)$. Each vertex has degree $y-1$, and is joined to $\binom{\frac{y-1}{2}}{2}$ other vertices in S. The result follows. □

We observe that $\partial_{(H_+ \cup H_{1+})}(S)$ consists entirely of edges joining pairs of wheels in H. Therefore, we can use the lemma to compute

$$|\partial_{(H_+ \cup H_{1+})}(S)| = 2p^r\left[\left(\frac{\frac{\phi(p^r)}{2} - 1}{2}\right)\left(\frac{\phi(p^r)}{2} - 1\right) - 2\binom{\frac{\frac{\phi(p^r)}{2}-1}{2}}{2}\right]$$

$$= \frac{1}{8}\left(p^{3r} - 2p^{3r-1} + p^{3r-2} - 4p^r\right).$$

To complete the computation note that:

$$|\partial_{(\pi_{p^r} - H)_1}(S)| \leq \frac{1}{2}\left(p^{3r-2} - p^{2r-1}\right)$$

$$|\partial_{(\pi_{p^r} - H)_-}(S)| = 0.$$

Putting everything together gives, for $p \equiv 3 \pmod 4$ and r a positive integer,

$$i(\pi_{p^r}) \leq \frac{p^{2r} - 2p^{2r-1} + 5p^{2r-2} - 4p^{r-1} + 4}{2\left(p^r + 2p^{r-1} - 3p^{r-2} + 4p^{-1}\right)}.$$

6. Concluding Remarks

The upper bounds obtained here for the Cheeger constants of the Platonic graphs lead immediately to corresponding bounds for the Cheeger constants of the Cayley graphs of $PSL(2, \mathbb{Z}_n)$ described earlier. In particular, since $i(G_{p^r}) \leq \frac{1}{p^r} i(\pi_{p^r})$, we have $i(G_{p^r}) < \frac{1}{2}$. If $\Gamma(N)$ is the congruence subgorup of level N of $SL(2, \mathbb{Z})$, then $\Gamma(N)$ acts by fractional linear transformations on the extended complex upper half plane $\mathcal{H}^*$. A fundamental domain for this action forms a Riemann surface, denoted $\mathcal{H}^*/\Gamma(N)$, that can be tesselated by modular triangles (i.e. the region in $\mathbb{C}^*$ above the unit circle and between the vertical lines with real parts $\pm\frac{1}{2}$). The graph having the triangles in this tesselation as vertices, with edges representing triangles sharing a common edge, is easily seen to be the Cayley graph G_n of $PSL(2, \mathbb{Z}_n)$.

Now, the isoperimetric number discussed here is a discretization of the Cheeger constant of a manifold used by spectral geometers to bound the eigenvalues of the Laplacian. In particular, if M is a closed, Riemannian manifold, its Cheeger constant, denoted by $h(M)$, is given by

$$h(M) = \inf_N \frac{\text{area } (N)}{\min(\text{vol } (A), \text{vol } (B))},$$

where N runs over all hypersurfaces dividing M into two pieces, A and B. Bounds on $i(G_n)$ lead immediately to bounds on $h(\Gamma_n)$, by the procedure described in [3]. Indeed, Buser introduced the isoperimetric number in [5] for precisely this reason.

However, Brooks has shown in [2] that one can not obtain sharp bounds on $h(M)$ by this procedure. In the case of $h(\mathcal{H}/\Gamma(N))$, Brooks and Zuk have shown in [4] that $h(\mathcal{H}/\Gamma(N)) \leq .441$, for all N. This bound is superior to what is attainable via the results presented here.

Finally, it is natural to wonder about the numbers $i(\pi_n)$ for composite n. In this case the vertices not contained in H are connected to each other in complex ways. Further, a more sophisticated indexing system will be required to handle these vertices. These facts pose significant challenges to extending the results presented in this paper.

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