LOWER BOUNDS ON THE CHEEGER CONSTANTS OF HIGHLY CONNECTED REGULAR GRAPHS

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ABSTRACT. We develop a method for obtaining lower bounds on the Cheeger constants of certain highly connected graphs. We then apply this technique to obtain new lower bounds on the Cheeger constants of two important families of graphs. Finally, we discuss the relevance of our bounds to determining the integrity of these graphs.

1. INTRODUCTION

Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. If $S \subseteq V(G)$ we define the boundary of S, denoted by ∂S, to be the set of edges having exactly one endpoint in S. We then define the Cheeger constant of G, denoted by $i(G)$, as

$$i(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum is taken over all sets S satisfying $|S| \leq \frac{1}{2}|V(G)|$. The quantity $i(G)$ is also called the isoperimetric number of G. The fraction $\frac{|\partial S|}{|S|}$ is called the isoperimetric quotient of S. A set S with minimal isoperimetric quotient is referred to as an isoperimetric set.

The Cheeger constant of a graph can be viewed as a measure of the graph's resiliency when viewed as a communications network. It finds a variety of applications in mathematics and computer science. For example, the eigenvalue spectrum and genus of the graph can be bounded in terms of the Cheeger constant, as described in [5] and [8].

In general it is difficult to compute the Cheeger constant of a graph explicitly. Consequently, in most cases one must be satisfied with deriving bounds on it. Upper bounds are easily obtained by evaluating the isoperimetric quotient of some strategically chosen set S. Non-trivial lower bounds, however, are far more difficult to come by.

In [2] Bollobás proved that almost every regular graph of degree k has isoperimetric number at least $(1-\eta)k/2$ for some explicit $0 < \eta < 1$ depending on k. He further showed that if we set

$$i(k) = \sup\{c \mid i(G) > c \text{ for infinitely many } k\text{-regular graphs } G\}$$

then $\lim_{k\to\infty} i(k)/k = 1/2$. Therefore, there are infinitely many k-regular graphs whose isoperimetric number is on the order of $k/2$. This is essentially the best possible, for large values of k.

In this paper we give a condition for k-regular graphs to have a large isoperimetric number. This allows us to give explicit examples of infinite families of k-regular graphs whose isoperimetric numbers are bounded away from zero. Under a further reasonable assumption, their isoperimetric numbers are essentially optimal in the sense of [2].

The technique employed in this paper is a significant refinement of a method first described in [3], and later used in [4]. To illustrate the idea, let G be k-regular and let $|V(G)| = n$. Let us further suppose that given any pair of vertices v_1 and v_2, there are precisely two length 2 paths joining them. Partition $V(G)$ into sets S_1 and S_2 with $|S_1| \leq |S_2|$. Each length 2 path joining a vertex in S_1 to a vertex in S_2 contributes a single edge to ∂S_1. Furthermore, we see that each edge appears in no more than $2(k-1)$ paths of length 2. This follows from the observation that each edge in ∂S_1 serves as the first edge of no more than $k-1$ such paths, and serves as the second of edge of no more than $k-1$ such paths. It follows that:

$$|\partial S_1| \geq \frac{2|S_1||S_2|}{2(k-1)}.$$

Since $|S_2| \geq \frac{n}{2}$, we obtain the bound

$$\frac{|\partial S_1|}{|S_1|} \geq \frac{n}{2(k-1)}.$$

Since the partition was arbitrary this gives a lower bound for $i(G)$.

Our improvement is based on the observation that the quantity $2(k-1)$ overestimates the number of paths of length 2 in which a given edge can reside. Note that the statement

$$i(G) = \inf_S \frac{|\partial S|}{|S|} \geq \frac{n}{2(k-1)}$$

implies that the average number of edges in ∂S_1 having v as an endpoint is $\frac{n}{2(k-1)}$. It follows that, on average, each edge in ∂S_1 lies in no more than $2k-1-\frac{n}{2(k-1)}$ such paths. This leads us to conclude that

$$i(G) \geq \frac{n}{2k-1-\frac{n}{2(k-1)}}.$$

This process can be iterated indefinitely. With each iteration we obtain an improved lower bound on $i(G)$.

This process will be formalized in Section 2. In Section 3 we apply our technique to two important families of graphs. Finally, in Section 4 we disucss the relevance of our results to determining the integrity of these graphs.

2. Improving the Bounds

Theorem 1. *Let G be a finite, simple, k-regular graph on n vertices. Let $v \in V(G)$ and let $m \in \mathbb{Z}^+$. Suppose that v is connected by at least r paths of length 2 to all but m other vertices in G. Finally, assume that $(2k-1)^2 > 2r(n-2m)$.*

Then

$$i(G) \geq \frac{2k-1-\sqrt{(2k-1)^2 - 2r\,(n-2m)}}{2}.$$

If in addition G has an isoperimetric set S satisfying $|S| = n/2$ and $k^2 > r(n-2m)$, then

$$i(G) \geq \frac{k-\sqrt{k^2 - r(n-2m)}}{2}.$$

Proof. Let G be a graph satisfying the given assumptions. Partition $V(G)$ into S_1 and S_2 with $|S_1| \leq |S_2|$. It follows that $|S_2| \geq n/2$.

Observe that there are at least $r|S_1|(|S_2|-m)$ paths of length 2 connecting vertices in S_1 to vertices in S_2. A single edge in each of those paths will be an element of ∂S_1. Furthermore, observe that each edge is an element of no more than $2(k-1)$ such paths. This implies that

$$|\partial S_1| \geq \frac{r|S_1|\,(|S_2|-m)}{2(k-1)}.$$

Consequently,

$$i(G) = \inf_S \frac{|\partial S|}{|S|} \geq \frac{r\,(|S_2|-m)}{2(k-1)} \geq \frac{\frac{rn}{2} - mr}{2(k-1)}.$$

To simplify the notation, set

$$i_1 = \frac{\frac{rn}{2} - mr}{2(k-1)}.$$

For $v \in S_1$ set

$$\partial v = \{e \in \partial S_1 \mid e \text{ is incident with } v\}.$$

Note that

$$\frac{\sum_{v\in S_1} |\partial v|}{|S_1|} = \frac{|\partial S_1|}{|S_1|} \geq i_1.$$

There are at least $r(|S_2|-m)$ paths of length 2 from v to S_2. One edge from each path will be in ∂v. Furthermore, note that each edge in ∂v is in no more than $k-1+k-|\partial v|$ distinct paths of length 2 containing v as a vertex. This follows from the observation that of the k edges incident with v, only $k-|\partial v|$ of them can serve as the first edge of such a length 2 path. It follows that

$$|\partial v| \geq \frac{r(|S_2|-m)}{2k-1-|\partial v|}.$$

Now, if $x_j > 0$ for $1 \leq j \leq n$, Lagrange's identity gives

$$\left(\sum_{j=1}^{n} x_j \right) \left(\sum_{j=1}^{n} \frac{1}{x_j} \right) \geq n^2.$$

We therefore obtain

$$\left(\sum_{v \in S_1} 2k - 1 - |\partial v| \right) \left(\sum_{v \in S_1} \frac{1}{2k - 1 - |\partial v|} \right) \geq |S_1|^2.$$

This implies

$$\begin{aligned} \sum_{v \in S_1} \frac{1}{2k-1-|\partial v|} &\geq \frac{|S_1|}{\frac{1}{|S_1|}\left(\sum_{v \in S_1} 2k - 1 - |\partial v| \right)} \\ &= \frac{|S_1|}{2k - 1 - \frac{1}{|S_1|} \sum_{v \in S_1} |\partial v|}. \end{aligned}$$

It follows that

$$\begin{aligned} |\partial S_1| = \sum_{v \in S_1} |\partial v| &\geq \sum_{v \in S_1} \frac{r(|S_2| - m)}{2k - 1 - |\partial v|} \\ &\geq \frac{r|S_1|(|S_2| - m)}{2k - 1 - \frac{1}{|S_1|} \sum_{v \in S_1} |\partial v|} \\ &\geq \frac{r|S_1|(|S_2| - m)}{2k - 1 - i_1}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|\partial S_1|}{|S_1|} &\geq \frac{r(|S_2| - m)}{2k - 1 - i_1} \\ &\geq \frac{\frac{rn}{2} - rm}{2k - 1 - i_1} = i_2. \end{aligned}$$

We conclude from this that $\frac{1}{|S_1|} \sum_{v \in S_1} |\partial v| \geq i_2$. Repeating the argument then gives us

$$\begin{aligned} \frac{|\partial S_1|}{|S_1|} &\geq \frac{\frac{rn}{2} - m}{2k - 1 - \frac{1}{|S_1|} \sum_{v \in S_1} |\partial v|} \\ &\geq \frac{\frac{rn}{2} - m}{2k - 1 - i_2} = i_3, \end{aligned}$$

and we can repeat this indefinitely.

Define

$$M(z) = \frac{\frac{rn}{2} - rm}{2k - 1 - z},$$

where $z \in \mathbb{C}$. Then $i(G) \geq M(1)$. Let $M^{\ell}(z)$ denote the ℓ^{th} iteration of this function. From the previous argument we have that

$$i(G) \geq \lim_{\ell \to \infty} M^{\ell}(1).$$

To determine this limit, note that $M(z)$ is a Mobius transformation whose fixed points are

$$z = \frac{2k - 1 \pm \sqrt{(2k-1)^2 - 2r\,(n - 2m)}}{2}.$$

It is shown in [1] that the iterates of such a Mobius transformation will converge to one of these fixed points. Set

$$z_0 = \frac{2k - 1 - \sqrt{(2k-1)^2 - 2r\,(n - 2m)}}{2}.$$

It is an easy calculation to show that

$$|M'(z_0)| < 1$$

and so z_0 is an attracting fixed point of $M(z)$. We conclude that

$$\lim_{\ell \to \infty} M^{\ell}(1) = z_0.$$

Consequently,

$$i(G) \geq z_0 = \frac{2k - 1 - \sqrt{(2k-1)^2 - 2r\,(n - 2m)}}{2},$$

as claimed.

For the second part, let G satisfy the given conditions. Let S_1, S_2 be a bipartition of $V(G)$ satisfying $|S_1| = |S_2|$. Arguing as before, we obtain

$$i(G) = \inf_{S} \frac{|\partial S|}{|S|} \geq \frac{r(|S_2| - m)}{2k - 1 - i_1}.$$

For $v \in S_1$ and $w \in S_2$ let

$$\partial vw = \{e \in \partial S_1 \mid e \text{ is incident with } v \text{ and } w\}.$$

Then $|\partial vw| = 1$ or 0, and $\sum_{w \in S_2} |\partial vw| = |\partial v|$.

There are at least r paths of length 2 containing both of the vertices v and w. Any edge in ∂vw is in no more than $k - |\partial v| + k - |\partial w|$ such paths of length 2. Thus

$$|\partial vw| \geq \frac{r}{2k - |\partial v| - |\partial w|}.$$

As before, we have the inequality

$$\frac{1}{|S_2|} \left(\sum_{w \in S_2} \frac{1}{2k - |\partial v| - |\partial w|} \right) \geq \frac{1}{2k - |\partial v| - \frac{1}{|S_2|} \sum_{w \in S_2} |\partial w|}.$$

It follows that

$$\frac{|\partial v|}{|S_2|} = \frac{1}{|S_2|} \sum_{w \in S_2} |\partial vw| \geq \frac{1}{|S_2|} \left(\sum_{w \in S_2} \frac{r}{2k - |\partial v| - |\partial w|} \right)$$
$$\geq \frac{r}{2k - |\partial v| - \frac{1}{|S_2|} \sum_{w \in S_2} |\partial w|}.$$

Now, let $S_2' \subset S_2$ contain those elements in S_2 joined by r paths of length 2 to the vertex v. Then we have $\frac{1}{|S_2|} \sum_{w \in S_2'} |\partial w| \geq \frac{|\partial S_2| - m}{|S_2|} \geq i_1$. This gives us

$$|\partial v| \geq \frac{r(|S_2| - m)}{2k - |\partial v| - i_1}$$

where i_1 is defined as before. Applying the same argument as in the first part of the theorem, we get

$$|\partial S_1| \geq \frac{r|S_1|(|S_2| - m)}{2k - 2i_1},$$

and therefore

$$i(G) \geq \frac{\frac{rn}{2} - rm}{2k - 2i_1}.$$

Define the Mobius transformation

$$M(z) = \frac{\frac{rn}{2} - rm}{2k - 2z}.$$

To determine $\lim_{\ell \to \infty} M^\ell(1)$, we note that the fixed points of this transformation are

$$z = \frac{k \pm \sqrt{k^2 - r(n - 2m)}}{2}.$$

Taking z_0 to be the smaller of these fixed points leads to the observation that

$$|M'(z_0)| < 1.$$

Therefore, the iterates of M converge to z_0, and the theorem is proved. □

3. Two Examples

We now consider two important families of graphs that satisfy the required conditions.

3.1. **Generalized Platonic Graphs.** Let $N > 1$ be an arbitrary positive integer. Let G be the set $\mathbb{Z}_N \times \mathbb{Z}_N - \{(0, 0)\}$. We define the graphs π_N as follows: The vertex set of π_N is given by $G/\{\pm 1\}$ and 2 vertices (a, b) and (c, d) are adjacent if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 \pmod{N}.$$

The expansion properties of these graphs have been studied previously, most notably in [3] and [6]. Their name derives from the fact that for small values of N the graphs π_N are isomorphic to the 1-skeletons of the familiar Platonic solids. We note in passing that the graphs π_N can also be obtained as quotients of certain Cayley graphs associated to the matrix group $PSL(2, \mathbb{Z}_N)$. As a result, their expansion properties are related to the Cheeger constants of certain Riemann surfaces of interest to number theorists.

The graphs π_N are N-regular and contain $\frac{N^2}{2}\prod_{p|N}\left(1-\frac{1}{p^2}\right)$ vertices. It is shown in [3] that if $(a,b),(c,d) \in V(\pi_N)$ are not multiples of each other, then there are precisely two paths of length two connecting them. We may therefore use our result from the previous section to derive a lower bound on $i(\pi_N)$.

Theorem 2. *Let $N > 1$ be an arbitrary positive integer and let π_N be the N^{th} Platonic graph defined above. Then*

$$i(\pi_N) \geq N - \frac{1}{2} - \frac{1}{2}\sqrt{4N^2 - 3 - 2N^2\prod_{p|N}\left(1 - \frac{1}{p^2}\right)}.$$

Proof. We apply the first inequality of Theorem 1 with $k = N$, $r = 2$ and $n = \frac{N^2}{2}\prod\left(1-\frac{1}{p^2}\right)$. To determine m, recall that vertices differing only by multiplication by -1 are identified. It follows that every vertex $(a,b) \in V(\pi_N)$ has no more than $(N+1)/2$ distinct, nonzero multiples. If S_1, S_2 partition $V(\pi_N)$, with $|S_1| \leq |S_2|$, then at most $\frac{N+1}{2} - 1$ of these multiples can be in S_2. It follows that we can set $m = \frac{N+1}{2} - 1$, and the result now follows immediately from Theorem 1. □

As explained in [6], there are strong reasons for believing that in the special case where p is a prime satisfying $p \equiv 1 \pmod 4$, and $\ell \in \mathbb{Z}^+$, then the graphs π_{p^ℓ} possess isoperimetric sets containing precisely half of the vertices. In this case we obtain the following result:

Theorem 3. *Let $p \equiv 1 \pmod 4$ and let $\ell \in \mathbb{Z}^+$. Assume that π_{p^ℓ} has an isoperimetric set S containing precisely half of the vertices. Then*

$$i(\pi_p) \geq \frac{p^\ell - \sqrt{p^{2\ell-2} + 2p^\ell - 6}}{2}.$$

Proof. The result follows immediately from applying the second inequality from Theorem 1 with $k = p^\ell$, $r = 2$, $n = \frac{p^{2\ell} - p^{2\ell-2}}{2}$ and $m = \frac{p^\ell - 1}{2} - 1$. □

In this case our bound approaches $p^\ell/2$ as $p \to \infty$. Note that the best known upper bounds on $i(\pi_{p^\ell})$, obtained in [6], also approach $p^\ell/2$ as $p \to \infty$.

3.2. **Quotients of Picard Group Cayley Graphs.** An important variation on the above example is obtained by considering the following set:

$$\Gamma_p = \{(\alpha,\beta) \mid \alpha,\beta \in \mathbb{Z}_p[i], (\alpha,\beta) \not\equiv (0,0) \pmod p\}/\langle \pm 1\rangle.$$

Here p is assumed to be a prime satisfying $p \equiv 3 \pmod 4$. We now construct a graph whose vertex set is Γ_p, with vertices (α,β) and (γ,δ) adjacent if and only if

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \pm 1, \pm i \pmod p.$$

We will denote this graph by G_p.

Let Γ denote the Picard group; i.e. the group of two by two matrices with Gaussian integer entries and determinant one. Let $\Gamma(p)$ denote the level p congruence subgroup of Γ (which consists of those matrices that reduce to the identity mod p). Then the graphs above can also be realized as certain quotients of Cayley graphs associated to $\Gamma/\Gamma(p)$. The expansion properties of these graphs are of interest because of their relationship to the eigenvalue spectrum of certain hyperbolic 3-manifolds. The details can be found in [9].

It is shown in [7] that if vertices (α,β) and (γ,δ) are not multiples of each other then they are joined by precisely eight paths of length 2. Also, it is easily seen that each vertex (α,β) has exactly $\frac{p^2-1}{2}$ distinct nonzero multiples in Γ_p. Furthermore, the graphs G_p are $2p^2$-regular and contain $\frac{p^4-1}{2}$ vertices.

Theorem 4. *Let p be a prime satisfying $p \equiv 3 \pmod 4$, and let G_p be the graph defined above. Then*

$$i(G_p) \geq 2p^2 - \frac{1}{2} - \frac{1}{2}\sqrt{8p^4 + 8p^2 - 39}.$$

Proof. We apply the first inequality of Theorem 1, with $k = 2p^2$, $n = \frac{p^4-1}{2}$, $m = \frac{p^2-1}{2} - 1$ and $r = 8$. The result follows immediately. □

As with the Platonic graphs, the results in [7] provide strong reasons for believing the graphs G_p possess isoperimetric sets containing exactly half the vertices. If this holds, then the second inequality of Theorem 1 gives:

Theorem 5. *Let G_p be the graphs defined above. Assume G_p has an isoperimetric set containing exactly half the vertices. Then*

$$i(G) \geq p^2 - \sqrt{2p^2 - 5}.$$

4. Applications to the Integrity of Graphs

Like the Cheeger constant, the integrity and edge-integrity are measures of graph vulnerability. Lower bounds on the Cheeger constant lead immediately to lower bounds on the integrity as well.

For $S \subset V(G)$ let $G - S$ be the graph obtained by deleting the vertices in S. Edges incident to any $v \in S$ are deleted as well. Let $m(G-S)$ denote the order of the largest component of $G - S$. Define the integrity of G to be

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\}.$$

In a similar fashion we define the edge-integrity to be

$$I'(G) = \min_{S \subset E(G)} \{|S| + m(G - S)\}.$$

Following the notation and terminology of [10], we say that a graph G on n vertices has the (α, β)-expanding property if, for any set A of at most αn vertices, $|\partial A| \geq \beta |A|$. Denote by $\mathrm{lb}(G)$ the lower bound on $i(G)$ obtained in Theorem 1. Then for any $0 < \alpha_0 \leq 1/2$ we have that G is an $(\alpha_0, \mathrm{lb}(G))$-expander.

The following theorem is a simple modification of Theorem 8 in [10]:

Theorem 6. *For a connected k-regular graph G on n vertices with the (α, β)-expanding property, we have*

$$I(G) \geq n \ \min\left(\alpha, \frac{\beta}{k+\beta}\right) \quad \textit{and} \quad I'(G) \geq n \ \min\left(\alpha, \frac{\beta}{2}\right).$$

Since $\mathrm{lb}(G) < k$, we have that $1/2 > \mathrm{lb}(G)/(k + \mathrm{lb}(G))$. For the graphs to which we are applying Theorem 1, we also have $\mathrm{lb}(G) \geq 1$. Consequently, we have

Theorem 7. *Let G be as in Theorem 1 and assume that $1 \leq \mathrm{lb}(G) \leq k$. Then*

$$I(G) \geq \frac{n \ \mathrm{lb}(G)}{k + \mathrm{lb}(G)} \quad \textit{and} \quad I'(G) \geq \frac{n}{2}.$$

.

Finally, using the bounds derived in the last section we have

Theorem 8. *Let π_{p^r} and G_p be as defined in the previous section. Then*

$$\begin{aligned}
I(\pi_{p^r}) &\geq \frac{\left(p^{2r} - p^{2r-2}\right)\left(p^r - \frac{1}{2} - \frac{1}{2}\sqrt{2p^{2r} + 2p^{2r-2} - 3}\right)}{4p^r - 1 - \sqrt{2p^{2r} + 2p^{2r-2} - 3}} \\
I'(\pi_{p^r}) &\geq \frac{p^{2r}}{4} - \frac{p^{2r-2}}{4} \\
I(G_p) &\geq \frac{(p^4 - 1)\left(p^2 - \frac{1}{4} - \frac{1}{4}\sqrt{8p^4 + 8p^2 - 39}\right)}{4p^2 - \frac{1}{2} - \frac{1}{2}\sqrt{8p^4 + 8p^2 - 39}} \\
I'(G_p) &\geq \frac{p^4 - 1}{4}.
\end{aligned}$$

Note that for large p, the lower bound for $I(\pi_{p^r})$ is on the order of $.11327 p^r$.

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