EXPANSION PROPERTIES OF LEVI GRAPHS

D. LANPHIER, C. MILLER, J. ROSENHOUSE, A. RUSSELL

ABSTRACT. The Levi graph of a balanced incomplete block design is the bipartite graph whose vertices are the points and blocks of the design, with each block adjacent to those points it contains. We derive upper and lower bounds on the isoperimetric numbers of such graphs, with particular attention to the special cases of finite projective planes and Hadamard designs.

1. INTRODUCTION

Denote the number of elements of a finite set S by $|S|$. For a finite, simple graph G we denote the vertex set and edge set by $V(G)$ and $E(G)$, respectively. Denote the subgraph induced by $S \subset V(G)$ by $G[S]$.

The following terminology and notation is taken from [1]. Let $\mathcal{X}$ be a finite set and let $\mathcal{A}$ be a collection of subsets of $\mathcal{X}$. We refer to the elements of $\mathcal{X}$ as points and the elements of $\mathcal{A}$ as blocks. We say the pair $(\mathcal{X}, \mathcal{A})$ is a balanced incomplete block design if there are nonnegative integers v, b, r, k and λ with $v > k > 0$ possessing the following properties: $v = |\mathcal{X}|$, $b = |\mathcal{A}|$, every point appears in r blocks, each block has k points, and every pair of points belongs to exactly λ blocks. For simplicity, we refer to balanced incomplete block designs as block designs. The block design is said to be linked (with link number μ) if $|B_1 \cap B_2| = \mu$ for any pair of distinct blocks $B_1, B_2 \in \mathcal{A}$.

A block design is said to be symmetric if $v = b$, or equivalently if $r = k$. It is well known (see [5]) that a block design is linked if and only if it is symmetric, and in this case we have $\lambda = \mu$. A symmetric block design with $\lambda = 1$ is called a finite projective plane. For any finite projective plane with n points, there must be a number q such that $n = q^2 + q + 1$. The number q is referred to as the order of the finite projective plane. It is also known that if $q = p^r$ for p a prime, then a finite projective plane of order q exists [12]. The existence of finite projective planes of non-prime-power order is an open question.

A Hadamard design of dimension n is a symmetric block design in which $v = 4n - 1$, $k = 2n - 1$ and $\lambda = n - 1$. The number $4n$ is called the order of the design. The name derives from the fact that a Hadamard design of order $4n$ exists if and only if a Hadamard matrix of order $4n$ exists (i.e. a $4n \times 4n$ matrix H with ± 1 entries satisfying $H^t H = nI$). See [12].

This research was supported by NSF grant number NSF-DMS 0243845.

It is possible to associate a bipartite graph G to an arbitrary block design as follows: The vertices of G are the points and blocks of the design. A point vertex is connected to a block vertex if the point lies in the block. The graph G is called the Levi graph of the block design. See [7]. The Levi graph associated to the Fano plane is shown in Figure One below. We note that among bipartite graphs with no cycles of length four, the Levi graphs of finite projective planes maximize the number of cycles of length six. See [2] and [6] for more information on the extremal properties of these and other graphs.

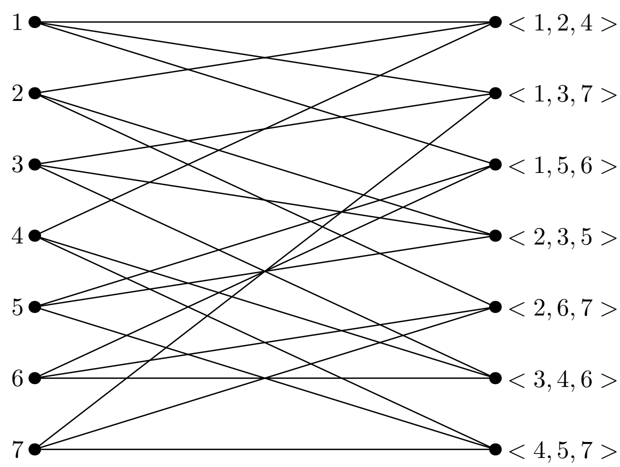

Figure One: Fano Plane Graph

Let G be a finite graph with vertex set $V(G)$. The boundary of $S \subset V(G)$, denoted ∂S, is the set of edges having exactly one endpoint in S. The isoperimetric number of G, denoted $i(G)$, is defined to be

$$i(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum is taken over all sets S satisfying $|S| \leq \frac{1}{2}|V(G)|$. The isoperimetric number is sometimes referred to as the expansion constant, the Cheeger constant, or the conductance of the graph. For $S \subset V(G)$ with $|S| \leq \frac{1}{2}|V(G)|$, we refer to the quantity $\frac{|\partial S|}{|S|}$ as the isoperimetric quotient of S and denote it by $i_S(G)$. If S is such that $i(G) = \frac{|\partial S|}{|S|}$ then S is referred to as an isoperimetric set for G. The problem of finding the isoperimetric number of a given graph is referred to as the isoperimetric problem. A variant on the isoperimetric problem for Levi graphs of projective planes was considered in [8].

In Section 2 we establish the main results, an upper bound on the isoperimetric numbers of general Levi graphs and a lower bound on the isoperimetric numbers of Levi graphs of symmetric block designs. It is a consequence of our results that the isoperimetric numbers of Levi graphs of finite projective planes and Hadamard designs are unbounded as their orders go to infinity. Since the vertex degrees of such Levi graphs are not fixed, they are not the best expanders in the sense of [11]. However, it is noted that since the vertex degree k of such a family of symmetric design Levi graphs increases far more slowly than the number of vertices v, the expansion properties of these graphs remain of interest.

2. BOUNDS FOR THE ISOPERIMETRIC NUMBERS

Note that if G is the Levi graph associated to a (v,b,r,k,λ) block design, then $|E(G)| = vr = bk$. This is easily seen by noting that every point vertex has degree r, while every block vertex has degree k. In the following let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the greatest integer and least integer functions, respectively. Our main results are the following two theorems:

Theorem 1. *Let G be the Levi graph associated to a (v,b,r,k,λ) block design. Then*

$$i(G) \leq \frac{\lceil \frac{b}{2} \rceil k}{\lfloor \frac{v}{2} \rfloor + \lceil \frac{b}{2} \rceil}.$$

Theorem 2. *Let G be the Levi graph associated to a symmetric block design with parameters (v,k,λ). Then*

$$i(G) \geq \frac{2\lambda^2(v-k)\binom{k}{2}}{3[(k-1)(k-2)(\lambda-1)^2 + \lambda^2(k-1)(v-k)]}$$

We establish the upper bounds for $i(G)$ by constructing a specific candidate isoperimetric set. In Section 3 we will see that in the special cases considered there, our upper bound is reasonably close to the actual value of $i(G)$. The lower bound is established by counting small cycles, following the methods in [3] and [10].

We will need the following lemma to prove Theorem 1:

Lemma 1. *Let G be the Levi graph of a (v,b,r,k,λ) block design. Let P be a set containing $\lfloor \frac{v}{2} \rfloor$ point vertices. Then there exists a set B of $\lceil \frac{b}{2} \rceil$ block vertices such that $|E(G[P \cup B])| \geq \lfloor \frac{v}{2} \rfloor \frac{r}{2}$.*

Proof. Let P be a set containing $\lfloor \frac{v}{2} \rfloor$ point vertices. There are $\lfloor \frac{v}{2} \rfloor r$ edges of G having an endpoint in P. Let B be an arbitrary set of $\lceil \frac{b}{2} \rceil$ block vertices and denote the complement of B by $\overline{B}$.

Every edge with one endvertex in P is connected either to an element of B or an element of $\overline{B}$. Therefore, it follows from the pigeonhole principle that either $E(G[P \cup B])$ or $E(G[P \cup \overline{B}])$ contains $\lfloor \frac{v}{2} \rfloor \frac{r}{2}$ elements. It follows that either B or $\overline{B}$ with one additional block vertex added will satisfy the conditions above. □

Proof. (*Theorem 1*) Let P be as in Lemma 1. Then there is a set of block vertices B satisfying $|B| = \lceil \frac{b}{2} \rceil$ and $|E(G[P \cup B])| \geq \lfloor \frac{v}{2} \rfloor \frac{r}{2}$. Set $S = P \cup B$. Then we have

$$|S| = \left\lfloor \frac{v}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil$$
$$|\partial S| \leq \left\lfloor \frac{v}{2} \right\rfloor r + \left\lceil \frac{b}{2} \right\rceil k - 2 \left\lfloor \frac{v}{2} \right\rfloor \frac{r}{2}.$$

Observe that since $|V(G)| = v + b$, we have that $|S| \leq \frac{1}{2}|V(G)|$. Since $i(G) \leq i_S(G)$, Theorem 1 follows. □

To obtain the lower bound we count the number of six-cycles containing arbitrary pairs of vertices in G. If G is bipartitioned into sets S_1 and S_2, then at least two edges out of each six-cycle containing vertices in opposite sets must be cut in order to disconnect the graph. To address overcounting, we determine the number of six-cycles in which a given edge appears. This is a variation of a method for establishing lower bounds on isoperimetric numbers that has been used previously (see [3], [4], [9] and [10], for example).

For a graph G, let $x, y \in V(G)$ and let $e \in E(G)$. We denote by $C_6(x,y)$ and $C_6(e)$ respectively, the number of six-cycles containing vertices x and y, and the number of six-cycles containing edge e. We begin with two lemmas.

Lemma 2. *Let G be the Levi graph of a symmetric block design with parameters (v,k,λ). Let p_1, p_2 be two distinct points and let b_1 be an arbitrary block satisfying $p_1 \notin b_1$. Then*

$$C_6(p_1, p_2) \geq C_6(p_1, b_1) = \lambda^2 \binom{k}{2}.$$

Throughout the following proof it will be convenient to make no distinction between points and blocks in the design, and the vertices representing those points and blocks in the Levi graph. We trust it will be clear from the context whether it is an element of the design or a vertex in the graph that is intended.

Proof. Let p_1, p_2 be arbitrary distinct point vertices. A six-cycle containing both of these points will contain exactly one further point vertex, which we denote by p_3. We distinguish two cases.

Assume that p_3 is chosen so there is no block containing all three points. There are $v-k$ such points. Each pair of point vertices appears in exactly λ blocks. Since no block contains all three points, we see that each of the three blocks in the cycle can be chosen in λ many ways. Consequently, there are a total of $\lambda^3(v-k)$ such six-cycles.

Now assume that p_3 is chosen so there is a block containing all three vertices. Let b_1 be a block containing p_1 and p_2. In a general symmetric block design, the number of blocks containing a given triple of points will depend on the particular triple. Since we seek a lower bound on $C_6(p_1, p_2)$, we assume there is only one such block. Consequently, p_3 must be chosen from among the $k-2$ remaining unused points in b_1.

If the six-cycle containing p_1, p_2, p_3 also contains the block b_1, then there will be $(\lambda-1)^2$ ways of choosing the remaining two blocks. Since there were λ many ways of choosing b_1, we have a total of $\lambda(\lambda-1)^2(k-2)$ six-cycles in this case.

If the six-cycle containing p_1, p_2, p_3 does not contain b_1, then the three block vertices can be chosen in $(\lambda-1)^3$ ways. This leads to $(\lambda-1)^3(k-2)$ six-cycles in this case.

Adding together our findings in each of these cases leads to

$$C_6(p_1, p_2) \geq \lambda^3(v-k) + (\lambda-1)^2(k-2) + (\lambda-1)^3(k-2).$$

To determine $C_6(p_1, b_1)$, note that the remaining point vertices in the cycle must be contained in b_1. Consequently, there will be $\binom{k}{2}$ ways of choosing those vertices. Having made those choices, there will be λ^2 many ways of choosing the remaining two block vertices.

It is a straightforward, but tedious, algebraic exercise to show that if p_1, p_2 and b_1 satisfy the assumptions of the lemma, then

$$\lambda^3(v-k) + (\lambda-1)^2(k-2) + (\lambda-1)^3(k-2) \geq C_6(p_1, b_1).$$

The lemma now follows. □

Let b_1, b_2 be arbitrary, distinct block vertices and let p_1, p_2 be arbitrary, distinct point vertices. We must now address the problem of determining $C_6(b_1, b_2)$. In a symmetric block design we have that $v = b$ and $\mu = \lambda$. Consequently, the counting argument provided for the determination of $C_6(p_1, p_2)$ goes through virtually unchanged for the determination of $C_6(b_1, b_2)$. We see, therefore, that $C_6(b_1, b_2) = C_6(p_1, p_2)$.

Lemma 3. *Let G be as above and let $e \in E(G)$. Then*

$$C_6(e) = (k-1)(k-2)(\lambda-1)^2 + \lambda^2(k-1)(v-k).$$

Proof. Denote by p_1 and b_1 the endpoints of the edge e. One of the remaining point vertices in the cycle must be chosen from the $k-1$ points of b_1 different from p_1. If the third point is likewise chosen from b_1 then there will be $k-2$ possible choices. There will then be $(\lambda-1)^2$ ways of choosing the remaining two blocks. If the third point is not chosen from b_1 then there will be $v-k$ possibilities, followed by λ^2 many ways of choosing the remaining blocks in the cycle. □

Proof. (*Theorem 2*) Assume that the vertices of G have been bipartitioned into sets S_1 and S_2, with $|S_1| \leq |S_2|$. By Lemma 2 there are at least $\lambda^2\binom{k}{2}$ six-cycles containing both v_1 and v_2. Each such cycle contributes at least two edges to ∂S_1. From Lemma 3 we have that each edge appears in no more than $(k-1)(k-2)(\lambda-1)^2 + \lambda^2(k-1)(v-k)$ six-cycles. Since there are three points on a given cycle non-adjacent to a given point, each cycle can be counted up to three times. It follows that

$$|\partial S_1| \geq \frac{2\lambda^2|S_1|(|S_2|-k)\binom{k}{2}}{3[(k-1)(k-2)(\lambda-1)^2+\lambda^2(k-1)(v-k)]}$$

Since $\frac{|\partial S_1|}{|S_1|}$ is an arbitrary isoperimetric quotient and $(|S_2|-k) \geq v-k$, the proof of Theorem 2 is complete. □

3. Projective Planes and Hadamard Designs

Two especially important classes of symmetric block designs are the finite projective planes and the Hadamard designs. They are united by the following observation (see [7]): Given a symmetric (v,k,λ) design, we must have

$$4n-1 \leq v \leq n^2+n+1,$$

where $n = k-\lambda$. The cases where $v = n^2+n+1$ are the projective planes and the cases where $v = 4n-1$ are the Hadamard designs.

Given their importance within the theory of block designs, we apply Theorems 1 and 2 to these cases and state them separately as corollaries. The closeness of the upper and lower bounds in these cases suggests that our bounds are fairly accurate.

Corollary 1. *Let G_q be the Levi graph of the finite projective plane of order q. Then*

$$\frac{q+1}{3} \leq i(G_q) \leq \frac{q+1}{2}\left(\frac{q^2+q}{q^2+q+1}\right)$$

Proof. The finite projective plane of order q has the parameters $v = q^2+q+1$, $k = q+1$ and $\lambda = 1$. □

We note in passing that the lower bound proven here was first presented, without proof, in [4].

Corollary 2. *Let G be the Levi graph of the Hadamard design with $v = 4n-1$. Then*

$$\left(\frac{(n-1)^2(2n-2)}{3n^2}\right)\left(\frac{2n-1}{4n-1}\right) \leq i(G) \leq 2n\left(\frac{2n-1}{4n-1}\right)$$

Proof. The Hadamard design with $v = 4n-1$ has the parameters $k = 2n-1$ and $\lambda = n-1$. □

References

[1] K. Bogart, *Introductory Combinatorics, 2nd. ed.*, Harcourt Brace Jovanovich, San Diego, 1990, pp. 327-381.
[2] B. Bollobas, *Extremal Graph Theory*, Academic Press, London, 1978, pp. 309-326.
[3] R. Brooks, P. Perry, P. Petersen, "On Cheeger's Inequality", *Comment. Math Helvetici*, Vol. 68 (1993), pp. 599-621.
[4] R. Brooks, A. Zuk, "On the Asymptotic Isoperimetric Constants for Riemann Surfaces and Graphs", *J. Differential Geom.*, Vol. 62 (2002), pp. 49-78.

[5] P. J. Cameron, J. H. Van Lint, *Designs, Graphs, Codes and their Links*, London Mathematical Society Student Texts, no. 22, Cambridge University Press, Cambridge, 1991, pp. 1-5.
[6] G. Fiorini, F. Lazebnik, "An Extremal Characterization of the Incidence Graphs of Projective Planes", *Acta Appl. Math.*, Vol 52 (1998), pp. 257-260.
[7] J. Gross, J. Yellen (eds.), *Handbook of Graph Theory*, CRC Press, Boca Raton, 2004. pp. 761-769.
[8] L. Harper, F. Hergert, "The Isoperimetric Problem in Finite Projective Planes", *Congr. Numer.*, Vol. 103 (1994), pp. 225-232.
[9] D. Lanphier, J. Rosenhouse, "A Decomposition Theorem for Cayley Graphs of Picard Group Quotients", *J. Combin. Math. Combin. Comput.*, Vol. 50 (2004), pp. 95-104.
[10] D. Lanphier, J. Rosenhouse, "Lower Bounds on the Cheeger Constants of Highly Connected Regular Graphs", *Congressus Numerantium*, Vol. 173 (2005), pp. 65-74.
[11] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progr. in Math. 125, Birkhäuser Verlag, 1994, pp. 1-6.
[12] K. H. Rosen, (ed.), *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, Boca Raton, 2000, pp. 753-796.

DOMINIC LANPHIER, DEPT. OF MATHEMATICS, WESTERN KENTUCKY UNIV., BOWLING GREEN, KY 42101, USA, DOMINIC.LANPHIER@WKU.EDU

CHRISTOPHER MILLER, DEPT. OF MATHEMATICS, FAIRFIELD UNIVERSITY, FAIRFIELD, CT 06824, USA, TUGBOAT730@YAHOO.COM

JASON ROSENHOUSE, DEPT. OF MATH. AND STAT., JAMES MADISON UNIV., HARRISONBURG, VA 22801, USA, ROSENHJD@JMU.EDU

AMBER RUSSELL, DEPT. OF MATH. AND STAT., MISSISSIPPI STATE UNIV., MISS. ST, MS 39762, USA, ACR31@MSSTATE.EDU