Math 431 — HW 4 — Due 04/29/09

You must turn in solutions to all the problems. The problem with your name is the one you will present in class.

1. **[10pts – Dana]**

   Definition (Separable)**.** Let F be a field, and let $f(x) \in F[x]$. We say that $f(x)$ is **separable** provided its irreducible factors over F have no repeated roots. If E/F is an extension of fields and $\alpha \in E$, we say that α is **separable** provided the minimal polynomial for α is separable. We say that E/F is a **separable extension** provided every element of E is separable over F.

   Let F be a field of characteristic zero. Let $f(x) \in F[x]$ be a monic irreducible polynomial over F. Show that $f(x)$ and $f'(x)$ cannot share any roots. Deduce that any extension E/F must be separable.

2. **[10pts – Adam]** Let L/F be a normal extension (using the Dummit and Foote definition of normal). Let M be an extension of F such that $F \subset M \subset L$. Suppose furthermore that $\sigma(M) = M$ for all $\sigma \in G = \mathrm{Gal}(L/F)$ and let a be an element of M which is not contained in any proper subfield of M. Prove that $p(x) = \prod_{\sigma \in G}(x - \sigma(a)) \in F[x]$, that $f(x) = \prod_{\sigma \in \mathrm{Gal}(M/F)}(x - \sigma(a))$ divides $p(x)$, and that $f(x)$ is the minimal polynomial for a.

3. **[10pts – Greg]** With $L/M/F$ as in the previous problem, let $\beta_1, \beta_2, \ldots, \beta_m$ be a basis for M over F. Let $f_1(x), f_2(x), \ldots, f_m(x)$ be the minimal polynomials for $\beta_1, \beta_2, \ldots, \beta_m$ respectively. Prove that M is the splitting field for $f(x) = \prod_i f_i(x)$.

4. **The Fundamental Theorem of Galois Theory** Let L/F be a separable extension with L the splitting field of a polynomial $p(x) \in F[x]$. Let $G = \mathrm{Gal}(L/F)$, and let E be a subfield of L containing F.

   (a) **[10pts – Emily]** E is the fixed field of some subgroup G_E of G.

   (b) **[10pts – Amanda]** E is a normal field extension of F if and only if G_E is a normal subgroup of G.

   (c) **[10pts – Kristin]** If E is normal, then $\mathrm{Gal}(E/F)$ is isomorphic to G/G_E. Furthermore, if K is any subextension of L containing F, then $[G : G_K]$ equals the degree of K/F.