Math 431 HW 2 Due 02/11/09

You must turn in solutions to all the problems. The problem with your name is the one you will present in class.

For this homework assignment, let V be a finite dimensional vector space over an algebraically complete field. Let L be a linear transformation on V.

1. **[10pts – Kristin]** Prove that there exists a positive integer n such that $\ker(L^n) \cap \operatorname{im}(L^n) = 0$. Show that for this value of n, $\ker(L^n) = \ker\left(L^{n+1}\right)$ and $\operatorname{im}(L^n) = \operatorname{im}\left(L^{n+1}\right)$. Show also that for this value of n, $V = \ker(L^n) \oplus \operatorname{im}(L^n)$. From here on, we will write L^∞ to denote L^n for this value of n.

2. **[10pts – Greg]** Let λ be an eigenvalue of L. Let V_λ be the λ-eigenspace. Note that $V_\lambda = \ker(L - \lambda I)$. Define $V_{(\lambda)} = \ker((L - \lambda I)^\infty)$. We call $V_{(\lambda)}$ the **generalized** λ-eigenspace of L. Prove that $V_{(\lambda)}$ is an L-invariant subspace.

3. **[10pts – Adam]** Prove that if μ and λ are both eigenvalues of L, then $V_{(\lambda)} \cap V_{(\mu)} = 0$. Conclude that the only eigenvalue of $L|_{V_{(\lambda)}}$ is λ.

4. **[10pts – Emily]** Let $\sigma(L)$ denote the set of all eigenvalues of L. Prove that

$$V = \bigoplus_{\lambda \in \sigma(L)} V_{(\lambda)} \oplus X$$

where X is L-invariant and $\sigma(L|_X) = \emptyset$.

5. **[10pts – Dana]** Prove that there exists a basis, β, for $V_{(\lambda)}$ which makes $\left[L|_{V_{(\lambda)}}\right]_\beta$ upper triangular. (Here $\left[L|_{V_{(\lambda)}}\right]_\beta$ denotes the matrix of $L|_{V_{(\lambda)}}$ with respect to β.) [Hint: Start with a basis for $V_\lambda = \ker(L - \lambda I)$, then extend this to a basis of $\ker\left((L - \lambda I)^2\right)$, then extend again...]

6. **[10pts – Amanda]** Use the above excercises to prove that there exists a basis of L which yields the Jordan Canonnical Form.