You must turn in solutions to all the problems. The problem with your name is the one you will present in class.

1. **[10pts – Greg]** Let $\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$ be a field with 4 elements. Write the addition and multiplication tables for this field (there is only one possibility for each). Explain your reasoning. Verify that $\mathbb{F}_4$ with these operations is a field.

2. **[10pts – Claire]** Find a polynomial $p(x) \in \mathbb{F}_2[x]$ such that $\mathbb{F}_4 \cong \mathbb{F}_2[x]/(p(x))$. Prove that your choice is correct. (Think back to Math 430.)

3. **[10pts – Dana]** How many solutions does the following system have in $\mathbb{F}_3$? What about in $\mathbb{F}_4$?

$$\begin{aligned} 2x_1 + x_2 + x_4 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

4. **[10pts – Amanda]** If A is an $n \times n$ matrix then the trace of A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

If C and D are $n \times n$ show that $tr(CD) = tr(DC)$ and conclude that if A and B are $n \times n$ and A is invertible then $\mathrm{tr}(A^{-1}BA) = tr(B)$.

5. **[10pts – Tom]** Prove, without appealing to the theory of row-reduced echelon form, that a system of homogeneous linear equations with more variables than equations has a nontrivial solution.

6. **[10pts – Kristin]** Let S be a system of homgeneous linear equations. Show that if $\vec{x}$ is a solution to S, then $\vec{x}$ will be a solution to any system that is obtained by performing a finite number of row operations on S.

7. **[10pts – Adam]** Let n be a positive integer, and let H_n be the matrix given by

$$H_n = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ & & \vdots & & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}.$$

Prove that H_n is invertible. [Hint: The only way I know how to do this problem involves integrals and much trickiness. Try to find your own solution, but don't be afraid to ask me for help.]

8. **[10pts – Emily]** Let $\mathbb{F}$ be a finite field, prove that $|\mathbb{F}|$ is a power of a prime.