

2 Probability

2.1 Sample Spaces and Events

Definition: “The **sample space** of an experiment, denoted by \mathcal{S} , is the set of all possible outcomes of that experiment.”

Example: Roll a (not necessarily fair) six-sided die once. The possible outcomes are the **faces** (i.e., dots) on the die.

$$\mathcal{S} = \left\{ \begin{array}{|c|} \hline \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \\ \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \\ \bullet \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \bullet \\ \bullet \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \bullet \\ \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \hline \end{array} \right\}$$

A different die might have six colors for the six sides (like a Rubik’s cube).



$$\mathcal{S} = \left\{ \begin{array}{|c|} \hline \color{red}\square \\ \hline \end{array}, \begin{array}{|c|} \hline \color{yellow}\square \\ \hline \end{array}, \begin{array}{|c|} \hline \color{green}\square \\ \hline \end{array}, \begin{array}{|c|} \hline \color{blue}\square \\ \hline \end{array}, \begin{array}{|c|} \hline \color{cyan}\square \\ \hline \end{array}, \begin{array}{|c|} \hline \color{magenta}\square \\ \hline \end{array} \right\}$$

□

Example: Toss a (not necessarily fair) coin once.

Example: Number of people living in a household.

Definition: An **event** is a subset of the sample space.

Example: For a (six-sided, numeric) die roll, name an **event**.

Additional definitions for events A and B :

(a) **union:** A or B , $A \cup B$

(b) **intersection:** A and B , $A \cap B$, AB

(c) **complement:** A'

(d) **mutually exclusive** or **disjoint:** $A \cap B = \emptyset$

A **Venn diagram** is a graphical representation of events.

Example: Let \mathcal{S} be the sample space of all Harrisonburg residents. let $A = \{\text{Person is female}\}$, and

let $B = \{\text{Person has high school diploma}\}$.

Example: Let \mathcal{S} be the sample space of all JMU students; let $A = \{\text{Person is a freshman}\}$, and let $B = \{\text{Person is a sophomore}\}$.

Example: Solve exercise 2.9, p. 51.

2.2 Axioms, Interpretations, and

Properties of Probabilities

Axiom 1: For any event A , $P(A) \geq 0$.

Axiom 2: $P(\mathcal{S}) = 1$.

Axiom 3: For A_1, A_2, A_3, \dots is an infinite collection of **disjoint** events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = \sum_{i=1}^{\infty} P(A_i).$$

Example: Consider **one** roll of any (unfair, numeric) **four**-sided die.

$$\mathcal{S} = \left\{ \begin{array}{|c|} \hline \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \\ \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \\ \bullet \\ \bullet \\ \hline \end{array}, \begin{array}{|c|} \hline \bullet \\ \bullet \\ \bullet \\ \bullet \\ \hline \end{array} \right\}$$

Remark: $0 \leq P(A) \leq 1$, and $P(A) \leq P(B)$ if $A \subseteq B$.

Example: Consider **one** roll of any (unfair, numeric) **four**-sided die.

Let $A = \left\{ \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \right\}$, and let

$$B = \left\{ \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \cup \begin{array}{|c|} \hline \bullet \\ \bullet \\ \hline \end{array} \right\}.$$

□

Interpreting Probability

What does it mean when we say that $P(\text{heads})=0.5$ for a coin? *Alternative question:* How do we find $P(\text{heads})$ for a coin?

The graphs below represent the sample proportion of heads, in tosses of a fair coin, for a large number of tosses.

More rigorous definition of *probability*, based on a long run proportion: A **probability** of an outcome is the proportion of times that the outcome is expected to occur when the experiment is repeated many times under identical conditions.

A **law of large numbers** states that a sample proportion, \hat{p} , “gets close to” the population proportion or probability, over a long run.

Example: What happens to the sample proportion of heads after many tosses of the coin?

Example: Consider a club consisting of 6 females and 4 males, and selections of club members are made WITH replacement. What happens to the sample proportion of females after many selections (WITH replacement)?

More Probability Properties

Same notation: Consider the sample space \mathcal{S} , and events A, B, A_1, A_2, \dots

Property: $P(A') = 1 - P(A)$

Proof:

Recall: If A and B are mutually exclusive, then $P(A \cap B) = 0$.

Example: Consider **one** roll of any (unfair, numeric) **four**-sided die.

Property: *Addition rule.* $P(A \cup B) = P(A) +$

$$P(B) - P(A \cap B)$$

Example: Suppose that in a city, 72% of the population got the flu shot, 33% of the population got the flu, and 18% of the population got the flu shot and the flu. Determine $P(\text{flu shot or flu})$.

□

Exercise 2.12, p. 57: *Slightly modified.*

Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a MasterCard.

Suppose that $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.22$.

The textbook uses 0.25 instead of 0.22.

- (a) Compute the probability that the selected individual has at least one of the two types of cards.
- (b) Compute the probability that the selected indi-

vidual has neither type of card.

- (c) Describe, in terms of A and B , the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.

□

Equally Likely Outcomes

If a sample space consists of N elementary outcomes such that each outcome is equally likely to be selected, then the probability of any particular outcome occurring is _____.

Example: Fair (*unbiased*) six-sided die.

Example: Fair (*unbiased*) coin.

Example: When rolling a fair die once, determine

$$P\left(\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \cup \begin{array}{|c|} \hline \bullet \\ \bullet \\ \hline \end{array}\right).$$

□

2.3 Counting Techniques

Example: Suppose we roll a **3-sided red** die once, and a **4-sided blue** die once.

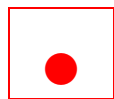
(a) List all possible pairs of outcomes.

(b) Determine the total number of outcomes.

(c) Assuming that the dice are fair, determine $P(\text{sum} = 4)$.

(c) Determine the total number of possible outcomes (where order matters) if we roll the **3-sided red** die **five** times and the **4-sided red** die **two** times.

“Where order matters” implies that rolling a

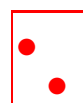


and a

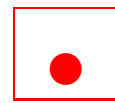


on the first and second

rolls differs from rolling a



and a



*on the first and second rolls of the **red** die,*

respectively.

Example: How many different types of pizzas can be made if **7** toppings are available?

Example: Suppose a club consists of **10** members. How many ways can we select one president, one chef, and one custodian, such that individuals **may** have more than one title?

Permutations

Example: Suppose a club consists of **10** members, and members may **NOT** hold more than one office.

- (a) How many ways can we select one president, one vice-president, and one secretary?
- (b) If Larry, Moe, and Curly are in this club, and officers are selected at random, what is the probability that Larry is president, Moe is vice-president, and Curly is secretary?

□

Determine $4!$, $5!$, $1!$, and $0!$.

In general, $P_{k,n} = \frac{n!}{(n-k)!}$

Note: With permutations, order **is** relevant.

Combinations

Example: Suppose a club consists of **10** members. How many ways can we select **3** members to serve on the charity committee?

Note: A committee consisting of persons (A, B, C) is identical to (A, C, B), (B, A, C), (B, C, A), (C, A, B), and (C, B, A).

In general, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

How many ways can we select **7** members **NOT** to serve on the charity committee?

Exercise 2.44, p. 67: Show that $\binom{n}{k} = \binom{n}{n-k}$.

Give an interpretation involving subsets.

□

- (c) Determine the probability of drawing **all diamonds** in a 5-card draw.
- (d) Determine the probability of drawing a **flush** in a 5-card draw.
- (e) Determine the probability of drawing a **royal flush** in a 5-card draw.

□

2.4 Conditional Probability

Example: (fictitious) Suppose that in Rhode Island, there are 100,000 college students. Among these 100,000 Rhode Island students, 10,000 attend (fictitious) Laplace-Fisher University and exactly half of the Rhode Island students are female. Among the 10,000 Laplace-Fisher students, 6,000 are female. Define the events A and B by $F = \{\text{Student is female}\}$ and $L = \{\text{Student attends Laplace-Fisher University}\}$

- (a) Determine the probability that a randomly se-

lected Rhode Island college student attends Laplace-Fisher University. In other words, what proportion of Rhode Island college students attend Laplace-Fisher University?

- (b) Determine the probability that a randomly selected Rhode Island college student attends Laplace-Fisher University AND is female.

In other words, what proportion of Rhode Island college students attend Laplace-Fisher University AND are female?

- (c) Determine the probability that a randomly selected Laplace-Fisher University student is female.

In other words, determine the probability that a randomly selected Rhode Island college student is female, **given** that the student attends Laplace-Fisher University.

In other words, determine the probability that a randomly selected Rhode Island college student is female, **conditional** that the student attends

Laplace-Fisher University.

In other words, what proportion of Laplace-Fisher students are female?

□

Definition: For events A and B , the **conditional probability** of event A , given that event B has occurred, is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$

This above definition is (*always true*), regardless of whether or not A and B are independent (to be defined in section 2.5).

Multiplication rule for conditional probabilities (*always true*): $P(A \text{ and } B) = P(A|B)P(B)$.
Similarly, $P(A \text{ and } B) = P(B|A)P(A)$.

Example: ♠♥♣♦ In a standard deck of 52 shuffled cards, determine the probability that the top card and bottom card are both diamonds. *No-*

tation: $T = \{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$.

The Law of Total Probability

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$

Example: Define the events

$G = \{\text{Individual has GENE linked to cancer}\}$, and

$C = \{\text{Individual gets CANCER}\}$. In a certain community, suppose that

20% of the people have this gene linked to cancer,

70% of the people WITH this gene will get cancer, and

10% of the people withOUT this gene will get cancer.

Determine the probability that a randomly selected individual will get cancer.

□

In general, we have equation (2.5): Let A_1, A_2, \dots, A_k

be mutually exclusive such that $\cup_{i=1}^k A_j = \mathcal{S}$. Then,

$$P(B) = \sum_{i=1}^k P(B \cap A_j) = \sum_{i=1}^k P(B|A_i)P(A_i).$$

Bayes' Theorem

Example: Define the events

$D = \{\text{Athlete uses DRUGS}\}$, and

$T = \{\text{Athlete TESTS positive for drugs}\}$.

For the athletes in a particular country, suppose that

10% of the athletes use drugs, **60%** of the drug-users test positive for drugs, and **1%** of the non-drug-users test positive for drugs.

- (a) State the probabilities in terms of the notation D and T .
- (b) What proportion of the athletes would test **positive** for drugs? Alternatively, what is the probability that a randomly selected athlete would test **positive** for drugs?
- (c) What proportion of the athletes would test **neg-**

- ative** for drugs? Alternatively, what is the probability that a randomly selected athlete would test **negative** for drugs?
- (d) What proportion of the **drug-users** would test **negative** for drugs? Alternatively, what is the probability that a randomly selected **drug-user** would test **negative** for drugs?
- (e) Given that an athlete tested **positive** for drugs, what is the likelihood that the athlete actually **uses** drugs?
- (f) Using a *tree diagram*, repeat part (e).
- (g) Given that an athlete tested **positive** for drugs, what is the likelihood that the athlete does **NOT use** drugs?

In general, **Bayes' theorem** is the following:

Let A_1, A_2, \dots, A_k be mutually exclusive such that

$\cup_{i=1}^k A_i = \mathcal{S}$. Then,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^k P(B|A_i)P(A_i)},$$

for $j = 1, \dots, k$.

Example: *Let's Make a Deal!* In this game, the contestant selects **one** door from among **three** doors. Behind one of the three doors is a car, and behind the other two doors are goats. Suppose the contestant selects door **#1**. Next, Monty Hall (the game-show host) opens a different door to **to intentionally show a goat**, and then offers the contestant the choice to switch doors.

Hence, if Monty Hall opens, say, door **#2**, should the contestant switch to door **#3** or keep door **#1**?

Define the events

$C_i = \{\text{Car is behind door } \#i\}$, and

$D_i = \{\text{Monty opens door } \#i\}$, for $i = 1, 2, 3$.

- (a) Determine the **prior** probabilities.
- (b) Determine the probability that Monty Hall opens door **#2**.
- (c) Determine the **posterior** probability that the car is behind door **#1**, given that Monty Hall opens door **#2**.
- (d) Determine the **posterior** probability that the car is behind door **#2**, given that Monty Hall opens door **#2**.
- (e) Determine the **posterior** probability that the car is behind door **#3**, given that Monty Hall opens door **#2**.
- (f) Recall that the contestant originally selected door **#1**. If Monty Hall opens door **#2** (to intentionally show a goat), show the contestant switch to door **#3** or keep door **#1**?
- (g) Rework parts (b), (c), (d), and (e) using a tree

diagram.

□

On exams you may use the tree diagram, or you may use the formulas directly. It is your choice!

2.5 Independence

Definition: Different trials of a random phenomenon are **independent** if the outcome of any one trial is not affected by the outcome of any other trial.

Example:

Definition: Trials which are not independent are called **dependent**.

Example:

Example: Tree diagram — Use a tree diagram to show the resulting **pairs** when a fair coin is tossed once and a fair **four**-sided die is rolled once.

Multiplication rule for independent events: If outcomes A and B are **independent**, then $P(A \text{ and } B) = P(A)P(B)$.

Example: Suppose that in a city, 72% of the population got the flu shot, 33% of the population got the flu, and 18% of the population got the flu shot and the flu. Are the events {Person gets the flu shot} and {Person gets the flu} independent?

Example: (*Know this example.*)

- (a) What is the likelihood of any particular airplane engine failing during flight?
- (b) If failure status of an engine is *independent* of all other engines, what is the likelihood that all

three engines fail in a three-engine plane?

- (c) What happened to a three-engine jet from Miami headed to Nassau, Bahamas (Eastern Air Lines, Flight 855, May 5, 1983)?

□

Definition: (Independence in terms of conditional probabilities) The following four statements are equivalent.

(a) Events A and B are **independent**.

(b) $P(A \text{ and } B) = P(A)P(B)$

(c) $P(A|B) = P(A)$

(d) $P(B|A) = P(B)$

Example: According to the American Red Cross,

Greensboro Chapter, 42% of Americans have type A blood, 85% of Americans have the Rh factor (positive), and 35.7% of Americans have type $A+$ blood.

Are the events {Person has type A blood} and {Person

has Rh factor} independent?

□

Example: *Revisit earlier example.* ♠♥♣♦

In a standard deck of 52 shuffled cards, let $T =$
 $\{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$

Determine the following probabilities and discuss whether
or not T and B are independent.

(a) $P(B)$

(b) $P(B|T)$

(c) $P(T)$

(d) $P(T|B)$

(e) $P(T \text{ and } B)$

(f) $P(T)P(B)$

□

Example: ♠♥♣♦ In a standard deck of 52 shuf-

fled cards, let

$F = \{\text{FIRST card drawn is a diamond}\}$ and let

$S = \{\text{SECOND card drawn is a diamond}\}.$

- (a) Determine the probability that the the first **two** cards drawn are diamonds, if the cards are drawn **WITH** replacement.
- (b) Determine the probability that the the first **two** cards drawn are diamonds, if the cards are drawn **withOUT** replacement.
- (c) Determine the probability that the the **second** card drawn is a diamond, if the cards are drawn **WITH** replacement.
- (c) Determine the probability that the the **second** card drawn is a diamond, if the cards are drawn **withOUT** replacement.

□

Example: A fair coin is tossed twice. Let $H_1 = \{\text{First toss is heads.}\}$, and $H_2 = \{\text{Second toss is heads.}\}$. Determine $P(H_2|H_1)$.

□

Remark: Events A_1, A_2, \dots, A_n can be **mutually independent** (p. 79).