

10 Analyzing the Association Between Categorical Variables

In this chapter, all observations are independent or nearly independent under the null hypothesis.

10.1 What is Independence and What is Association?

Comparing Percentages

Example: The *Titanic* collided with an iceberg April 14, 1912.

The data below are from the *British Board of Trade Inquiry Report* (1990), written originally on July 30, 1912. Did the different classes of passengers have equal chances of survival?

Observed table	First	Second	Third	Crew	total
Alive	203	118	178	212	711
Dead	122	167	528	673	1490
total	325	285	706	885	2201

Determine the **conditional probabilities** of **survival, given** the **class** of the passenger.

- (a) Determine the probability that a randomly selected passenger **survived, given** that the passenger was **first**-class. Alternatively, determine the proportion of **first**-class passengers who **survived**.

Let $S = \{\text{Passenger survived}\}$ and

$D = \{\text{Passenger died}\}$.

- (b) Determine the probability that a randomly selected passenger **survived, given** that the passenger was **second**-class. Alternatively, determine the proportion of **second**-class passengers

who **survived**.

- (c) Determine the probability that a randomly selected passenger **survived**, **given** that the passenger was **third**-class. Alternatively, determine the proportion of **third**-class passengers who **survived**.
- (d) Determine the probability that a randomly selected passenger **survived**, **given** that the passenger was a member of the **crew**. Alternatively, determine the proportion of **crew**-members who **survived**.
- (e) Do the **class** of the passenger and the **survival** status seem to be **independent** or **dependent**? In other words, do the discrepancies among the answers to parts (a), (b), (c), and (d) seem to be caused by **random chance**, or do the discrepancies seem to be caused by discrimination among the four **classes** of passen-

gers?

- (f) Determine the probability that a randomly selected passenger **died**, **given** that the passenger was **first**-class. Alternatively, determine the proportion of **first**-class passengers who **died**.
- (g) Determine the probability that a randomly selected passenger **survived**. Alternatively, determine the proportion of passengers who **survived**.
- (h) Determine the probability that a randomly selected passenger **died**. Alternatively, determine the proportion of passengers who **died**.

□

10.2 How Can We Test Whether Categorical Variables Are Independent?

What Do We Expect for Cell Counts If the Variables Are Independent?

Example: *Revisit the Titanic.*

- (a) How many **first-class** passengers do we **expect** to have **survived** if the **class** and the **survival status** of the passengers were **independent**?
- (b) How many **first-class** passengers do we **expect** to have **died** if the **class** and the **survival status** of the passengers were **independent**?
- (c) How many **second-class** passengers do we **expect** to have **survived** if the **class** and the **survival status** of the passengers were **independent**?
- (d) What is the formula for the **expected count** under the assumption of **independence** between the **class** and the **survival status** of the pas-

sengers?

- (e) Complete the table below for **expected counts** under H_0 (i.e., “The class and survival status of the passengers were **independent**.”).

Expected table					
under H_0	First	Second	Third	Crew	total
Alive					
Dead					
total					

□

The **chi-squared statistic** is defined as

$$X^2 = \sum \frac{(\text{observed count} - \text{expected count})^2}{\text{expected count}}$$

The statistic X^2 has a distribution which is approximated **chi-squared**, denoted χ^2 , with **degrees of freedom =**

$$[(\text{Number of rows}) - 1] \times [(\text{Number of columns}) - 1],$$

if the **expected count** is at least **5** in all cells

under the assumption that the two variables are **independent**.

Below are the probability density functions of the χ^2 distribution with degrees of freedom equal to **1, 2, 3, 4, 5, 6, 7, 8, and 9**.

Example:

- (a) Compute $P(\chi_4^2 > 6.6)$.
- (b) Compute $P(\chi_4^2 > 3.8)$.

Chi-squared table, p. A4							
	Right-Tail Probability						
df	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.83
2	2.77	4.61	5.99	7.38	9.21	10.60	13.82
3	4.11	6.25	7.81	9.35	11.34	12.84	16.27
4	5.39	7.78	9.49	11.14	13.28	14.86	18.47
5	6.63	9.24	11.07	12.83	15.09	16.75	20.52
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

□

Example: *Revisit the Titanic.* Test at level 0.01 if the **class** and the **survival status** of the passengers were **dependent**, according to the fol-

lowing steps.

- (a) State the null and alternative hypotheses.
- (b) Determine the value of the **test statistic**.

$$\begin{aligned}
 \chi^2 &= \sum \frac{(\text{observed count} - \text{expected count})^2}{\text{expected count}} = \frac{(203-105.0)^2}{105.0} \\
 &+ \frac{(118-92.1)^2}{92.1} + \frac{(178-228.1)^2}{228.1} + \frac{(212-285.9)^2}{285.9} + \frac{(122-220.0)^2}{220.0} \\
 &+ \frac{(167-192.9)^2}{192.9} + \frac{(528-477.9)^2}{477.9} + \frac{(673-599.1)^2}{599.1} \\
 &= 91.50 + 7.31 + 10.99 + 19.10 + 43.66 + 3.49 + \\
 &5.24 + 9.11 = 190.4
 \end{aligned}$$

- (c) Determine the **degrees of freedom**.
- (d) Is the chi-squared approximation valid?
- (e) Determine the *P*-value.

Chi-squared table, p. A4

<i>df</i>	Right-Tail Probability						
	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.83
2	2.77	4.61	5.99	7.38	9.21	10.60	13.82
3	4.11	6.25	7.81	9.35	11.34	12.84	16.27
4	5.39	7.78	9.49	11.14	13.28	14.86	18.47
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- (f) State the conclusion in statistical terms and in regular English.

□

Remark: The **rows** and **columns** are interchangeable.

Remark: The above test often is called a **chi-square test for independence**.

Remark: If one of the variables is **explanatory** and the other variable is **response**, then this test also may be called a **chi-square test for homogeneity**; i.e., are the distributions homogeneous or heterogeneous?

Example: *Revisit the Titanic.* Restate the null and alternative hypotheses in the context of a **test for homogeneity**.

□

Example: Are **tattoos** and **hepatitis C** infections **dependent**? Test at level $\alpha = 0.01$.

Hepatitis C is a potentially fatal disease that attacks

the liver, and causes 10,000 to 20,000 deaths in the U.S. each year from cirrhosis and liver cancer. Hepatitis C affects an estimated 4 million people in the U.S. The following data are based on a study in the University of Texas Southwestern Medical Center at Dallas. In 1991–1992, 626 participants in the study were patients of an orthopedic spinal clinic, a setting that provided a large volume of patients seeing a physician for reasons unrelated to blood-born infection. Participants unaware of their hepatitis status were examined, interviewed for risk factors, and tested for hepatitis C. Below is the **observed** table.

Population #	status	hepatitis C	no hepatitis C	total
1	tattoo	25 ()	88 ()	113
2	no tattoo	22 ()	491 ()	513
	total	47	579	626

- (a) Determine the proportion of people **with** tattoos who **have** hepatitis C.
- (b) Determine the proportion of people **without** tattoos who **have** hepatitis C.
- (c) State the null and alternative hypotheses.
- (d) Determine the **expected count** of people with both tattoos and hepatitis C under H_0 .
- (e) In the above table, list the **expected counts** in the parentheses.
- (f) Determine the value of the **test statistic**.
- (g) Determine the **degrees of freedom**.

- (h) Is the chi-squared approximation valid?
- (i) Determine the P -value.

Chi-squared table, p. A4							
	Right-Tail Probability						
df	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.83
2	2.77	4.61	5.99	7.38	9.21	10.60	13.82
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4	5.39	7.78	9.49	11.14	13.28	14.86	18.47
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- (j) State the conclusion in statistical terms and in regular English.
- (k) Restate the hypotheses in the context of a **test for homogeneity**.

H_0 : The two populations, those WITH tattoos and those withOUT tattoos, are **homogeneous** in terms of infection rates of hepatitis C.

H_a : The two populations, those who have tattoos and those who do not have tattoos, are **heterogeneous** in terms of infection rates of hepatitis C.

□

Remark: Association does NOT imply causation.

Remark: For a 2×2 table, the χ^2 test for independence (or homogeneity) produces the same P -value and same conclusion as a two-sided Z -test on the difference between two proportions, since $Z^2 = \chi_1^2$.

Example: *Revisit tattoos and hepatitis C.* Are **tattoos** and **hepatitis C** infections **dependent**? Test at level $\alpha = 0.01$ using the Z -test on the difference between two proportions.

Population #	status	hepatitis C	no hepatitis C	total
1	tattoo	25	88	113
2	no tattoo	22	491	513
	total	47	579	626

(a) Define your notation.

Let p_1 be the *unknown* **population** proportion of people **with tattoos** who **have**

hepatitis C.

Alternatively, let p_1 be the *unknown population* proportion of **tattooed** people who **have hepatitis C.**

Let p_2 be the *unknown population* proportion of people **withOUT** tattoos who **have hepatitis C.**

Alternatively, let p_2 be the *unknown population* proportion of **NON-tattooed** people who **have hepatitis C.**

- (b) State the null and alternative hypotheses in terms of your notation.
- (c) Define your notation for the **samples.**

Let \hat{p}_1 be the **sample** proportion of people **with tattoos** who **have hepatitis C.**

Alternatively, let \hat{p}_1 be the **sample** proportion of **tattooed** people who **have hepatitis C.**

Let \hat{p}_2 be the **sample** proportion of people

withOUT tattoos who **have** hepatitis C.

Alternatively, let \hat{p}_2 be the **sample** proportion of **NON-tattooed** people who **have** hepatitis C.

Let \hat{p} be the **overall sample** proportion of **people** who **have** hepatitis C.

- (d) Evaluate \hat{p}_1 , \hat{p}_2 , and \hat{p} .
- (e) Determine the **point estimate** of $(p_1 - p_2)$.
- (f) Interpret the above **point estimate** in regular English.

We estimate that for 17.8% of patients, having a tattoo results in having hepatitis C instead of not having hepatitis C.

- (g) Check the rule of thumb.
- (h) Determine the value of the **standardized test statistic**.
- (i) Determine the P -value.

Standard normal table, pp. A1–A2										
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(j) State the conclusion in statistical terms and in regular English.

We conclude that the **population** proportion of people **with tattoos** who **have hepatitis C differs** from the **population** proportion of people **withOUT** tattoos who **have hepatitis C**.

□

Remark: The χ^2 test does **NOT** provide results for **one**-sided tests, whereas the **Z** test on the difference between two proportions **DOES** provide results for **one**-sided tests.

10.S Supplement: How Can We Test for the Population of a Categorical Variable?

H_0 : The data are from a specified given population.

H_a : The data are from a different population.

The test often is called a **goodness-of-fit** test, since we are testing if the specified population provides a good fit to the data.

Example: *Hypothetical data.* Are the three sections of Spanish 101 (all taught at the same time) equally likely to be selected by the students? Test at level 0.05.

	Section #		
	1	2	3
Frequency	137	156	109

(a) Define your notation.

Let p_1 be the probability that a Spanish 101 student will enroll in Section #1 (or the population proportion of all Spanish 101 students who would enroll in Section #1).

Let p_2 be the probability that a Spanish 101 student will enroll in Section #2.

Let p_3 be the probability that a Spanish 101 student will enroll in Section #3.

- (b) State the null and alternative hypotheses.
- (c) Determine the **expected count** of students in Section #1 under H_0 .
- (d) Determine the value of the **test statistic**.
- (e) Determine the **degrees of freedom**.
- (f) Is the chi-squared approximation valid?
- (g) Determine the P -value.

Chi-squared table, p. A4

df	Right-Tail Probability						
	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.83
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3	4.11	6.25	7.81	9.35	11.34	12.84	16.27
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(h) State the conclusion in statistical terms and in regular English.

□

Example: The article “Linkage Studies of the Tomato” (*Transactions of the Royal Canadian Institute* [1931]: 1–19) reported the accompanying data on phenotypes resulting from crossing tall cut-leaf tomatoes with dwarf potato-leaf tomatoes. There are four possible phenotypes: (1) tall cut-leaf, (2) tall potato leaf, (3) dwarf cut-leaf, and (4) dwarf potato-leaf. Mendel’s laws of inheritance imply that the population proportions should be $9/16$, $3/16$, $3/16$, and $1/16$ for groups 1, 2, 3, and 4, respectively.

Test at level 0.05 if these data below are consistent with Mendel's laws of inheritance.

	Phenotype			
	1	2	3	4
Frequency	926	288	293	104

(a) Define your notation.

Let p_1 be the probability that the tomato will be in group #1.

Let p_2 be the probability that the tomato will be in group #2.

Let p_3 be the probability that the tomato will be in group #3.

Let p_4 be the probability that the tomato will be in group #4.

(b) State the null and alternative hypotheses.

(c) Determine the **expected count** of tomatoes in each of the four groups under H_0 .

- (d) Determine the value of the **test statistic**.
- (e) Determine the **degrees of freedom**.
- (f) Is the chi-squared approximation valid?
- (g) Determine the *P*-value.

Chi-squared table, p. A4							
	Right-Tail Probability						
<i>df</i>	0.250	0.100	0.050	0.025	0.010	0.005	0.001
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4	5.39	7.78	9.49	11.14	13.28	14.86	18.47
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- (h) State the conclusion in statistical terms and in regular English.

□

Example: *How accurate were Germany’s flying-bombs?* During World War II, Germany sent 537 flying-bombs in an area consisting of 144 km² of Southern London. This area was subdivided into 576 square **regions** of equal size. The table below summarizes the data. Did Germany drop the

bombs in random **regions** of Southern London, or was Germany targeting specific **regions**? Test at level 0.05.

Number (x) of bomb hits per area	Number of regions with x bomb hits
0	229
1	211
2	93
3	35
4 or more	8
sum	576

- (a) State the null and alternative hypotheses.
- (b) Letting x be the number of flying-bombs to hit any one particular **region** of Southern London, show that x has the following probability distribution under H_0 .

x	$P(x)$
0	0.3933
1	0.3673
2	0.1712
3	0.0531
4 or more	0.0150
sum	1

- (c) Determine the **expected** number of **regions** to receive 0 bomb hits, 1 hit, 2 hits, 3 hits, and 4 or more hits, under H_0 .
- (d) Determine the value of the **test statistic**.
- (e) Determine the **degrees of freedom**.
- (f) Is the chi-squared approximation valid?
- (g) Determine the P -value.

Chi-squared table, p. A4

<i>df</i>	Right-Tail Probability						
	0.250	0.100	0.050	0.025	0.010	0.005	0.001
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5	6.63	9.24	11.07	12.83	15.09	16.75	20.52
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

(h) State the conclusion in statistical terms and in regular English.

□