APPROXIMATING IMPROPER DEFINITE INTEGRALS

OPTIONAL SOLO Last @!!@*! Programming Assignment, No. 8 (20 points)

Math 248 Computers and Numerical Algorithms–Pruett [1]

DATE ASSIGNED: Tuesday, 2 December, 2008 **DATE DUE:** Monday, 8 December, 2008
REFERENCE: *Calculus* by Anton, pp. 485-489, or by Salas et al., pp. 625-633.

BACKGROUND: The following convergent *improper* definite integral I arises in statistics:

$$I = \int_0^\infty e^{-x^2} dx \tag{1}$$

Unfortunately, the integrand above has *no elementary antiderivative*; hence, the integral must be evaluated numerically. However, from a naive point of view, it would seem that we have no hope of computing the integral because of the infinite upper limit. Never fear, cleverness to the rescue!

ASSIGNMENT: Evaluate Eq. 1 numerically to machine single precision. The following step-wise approach is recommended to avoid end-of-semester panic:

1. Sketch the graph of the integrand. Notice that it gets very small very quickly for large x. Now split the integral into "head" and "tail" parts as follows:

$$I = \int_0^\infty e^{-x^2} dx = H + T \tag{2}$$

   where

$$\begin{aligned} H(c) &= \int_0^c e^{-x^2} dx &(3)\\ T(c) &= \int_c^\infty e^{-x^2} dx &(4) \end{aligned}$$

   and c is the "cutoff" to be determined. In your sketch, shade (differently) the areas representing the head and tail of the integral (for some arbitrary value of $c > 1$).

2. For $x > 1$, show that e^{-x^2} is bounded above by xe^{-x^2}. For $c > 1$, deduce that

$$\int_c^\infty e^{-x^2} dx < \int_c^\infty xe^{-x^2} dx \tag{5}$$

   The improper integral on the right-hand side above can be evaluated exactly by antidifferentiation and by taking an appropriate limit. (Refer to *Calculus* by Anton, Section 10.1.) Evaluate the integral for arbitrary c, then determine a value of c for which $|T| < 0.5 \times 10^{-7}$.

3. Write a *simple* FORTAN 90 program to approximate $H(c)$ by numerical integration, for the particular value of the cutoff c determined in the previous step. Let N be the numerical approximation to H and let E be its truncation error; that is, H = N + E. Thus

$$I = H + T = (N + E) + T \Rightarrow I - N = E + T \tag{6}$$

4. Choose a value of h for your numerical approximation so that $|E| < 0.5 \times 10^{-7}$, and run your code for this value. Provided you have done everything right, $|E + T| < 10^{-7}$ and your result N is correct to at least 7 digits. Celebrate!

As always, you are allowed to discuss programming issues with other students, but the actual coding of your program should be accomplished individually, and your program should be unique.

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[1] Thanks to Dr. Carter Lyons for the suggestion.