Linear Recurrence Relations in Music

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Linear Recurrence Relation

Each term of a sequence defined by a linear combination of the previous terms

For example,

$3, 4, 2, 7, 7, 19, 26, \ldots \qquad a_n = a_{n-1} + 2a_{n-2} - a_{n-3}$

$5, -4, -3, 0, 1, 7, -12, 1, \ldots \qquad a_n = a_{n-4} + a_{n-5} - a_{n-6}$

Linear Recurrence Relation

In general,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

is a linear recurrence relation of order-d.

The sequence is determined uniquely with initial values $a_0, \ldots, a_{d-1}$.

How to find a Linear Recurrence Relation

Given a sequence of numbers $a_0, a_1, \ldots a_N$, we want to find the smallest ordered linear recurrence relation that generates $a_0, a_1, \ldots, a_N$.

Suppose $d = 3$. Then we need

$$\begin{aligned} a_3 &= c_1 a_2 + c_2 a_1 + c_3 a_0 \\ a_4 &= c_1 a_3 + c_2 a_2 + c_3 a_1 \\ &\vdots \\ a_N &= c_1 a_{N-1} + c_2 a_{N-2} + c_3 a_{N-3} \end{aligned}$$

How to find a Linear Recurrence Relation

Coefficients found by solving

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ & \vdots & \\ a_{N-3} & a_{N-2} & a_{N-1} \end{bmatrix} \begin{bmatrix} c_3 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_N \end{bmatrix}$$

Berlekamp-Massey Algorithm (1969)

- One-pass algorithm to find the smallest-ordered linear recurrence relation
- Works over an infinite field or finite field

Linear Recurrence Relations in Music

- Only interested in melodic progression, not rhythm
- Data in standard MIDI format and shifted to appropriate field
- Usually working over $\mathbb{Z}_p$ (p prime) where p depends on the melody range

Some Examples

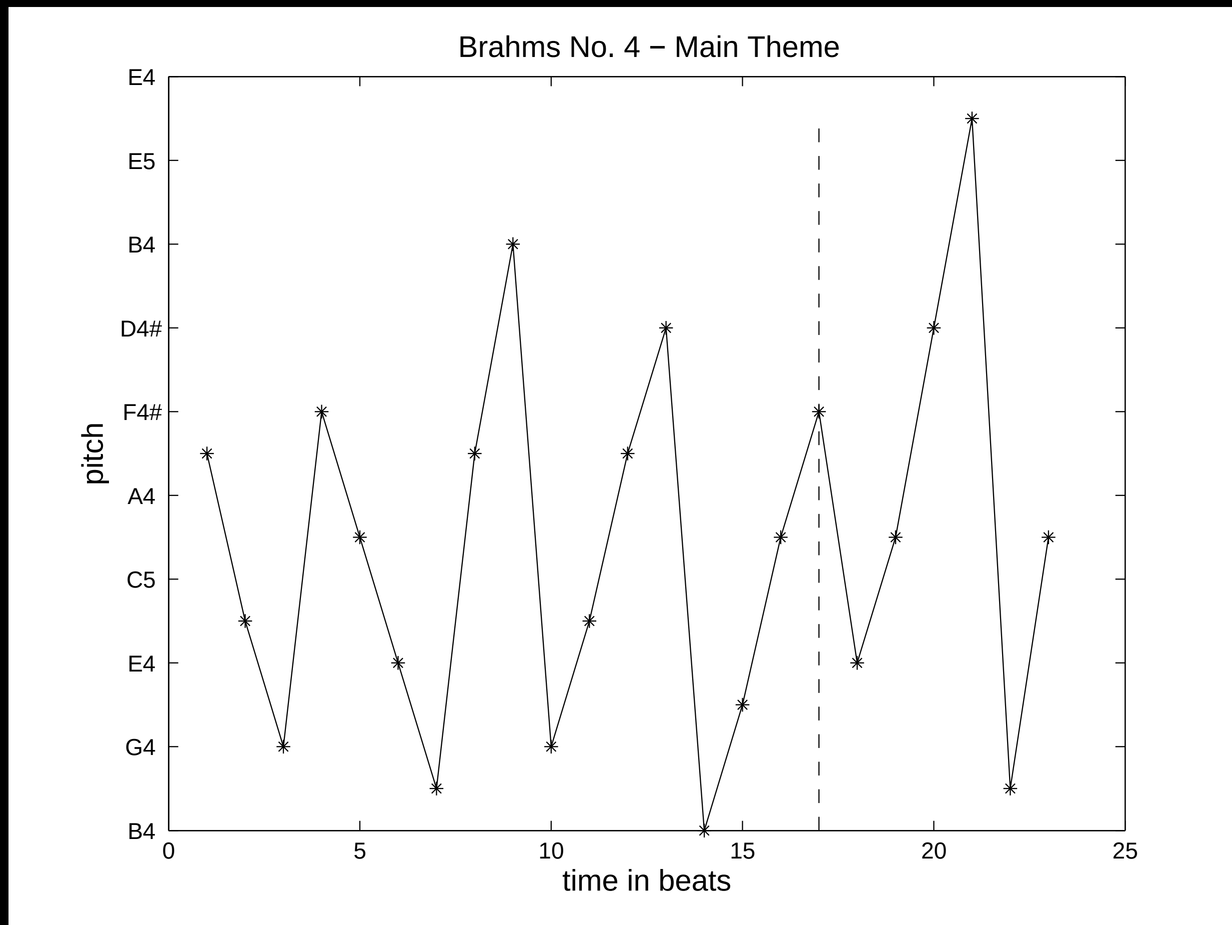
Brahms Symphony No.4

Main Theme: 17 notes, used $\mathbb{Z}_{23}$

8th-order recurrence relation found:

$$\begin{aligned} a_9 &= a_8 + 13a_7 + 17a_6 + 15a_5 + 18a_4 + 15a_3 \\ &+ 18a_2 + 16a_1 + 0a_0 \end{aligned}$$

Brahms Symphony No.4



J.S. Bach, Invention No. 13

Two possible phrases: 11 notes or 21 notes ($\mathbb{Z}_{13}$)

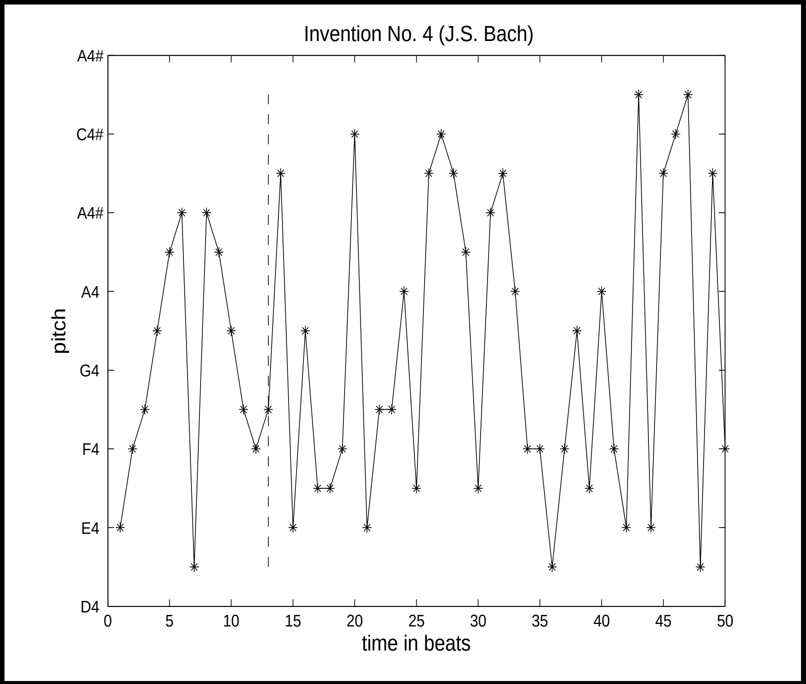
- Linear recurrence relations found
- Generated sequences seem to stay in same key signature

Can this recurrence relation be used to generate ‘Bach-like’ melody lines?

J.S. Bach, Invention No. 4

13 note phrase ($\mathbb{Z}_{13}$)

- 6th-order linear recurrence relation
- Not necessarily aesthetically pleasing...



Beethoven Piano Sonata No. 8 "Pathetique"

Short phrase: 8 notes
Long phrase: 42 notes

- Linear recurrence relations found $(\mathbb{Z}_{13})$
- Cycle repeats (period=42) for short-note phrase

More Bach...

WTC I Fugue No. 4	2nd-order RR ($\mathbb{Z}$)
WTC I Fugue No. 7	12th-order RR ($\mathbb{Z}_{19}$)
WTC I Prelude No. 11	24th-order RR ($\mathbb{Z}_{17}$)

Linear Recurrence Relations over Rings

- Perhaps more logical to use $\mathbb{Z}_{12}, \mathbb{Z}_{24}, \ldots$
- Does not find an exact linear recurrence relation
- Typically melodies generated cycle amongst the last few notes in the phrase

Your turn!

- Choose field
- Choose starting sequence
- Choose coefficients of linear recurrence relation

Can you compose music with linear recurrence relations?