# The Rank of a Tensor

## *Joint Mathematics Meetings*
## *January 5, 2007*

Carla D. Martin

Department of Mathematics and Statistics
James Madison University

# The Rank of a Matrix

Every matrix can be written as a sum of rank-1 matrices:

$$
\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= XY^T \\
&= \left[\, x_1 \,\middle|\, x_2 \,\right] \left[\, y_1 \,\middle|\, y_2 \,\right]^T \\
&= x_1 y_1^T + x_2 y_2^T \\
&= (x_1 \circ y_1) + (x_2 \circ y_2)
\end{aligned}
$$

A full-rank matrix is generated with probability one

# Special Case: Matrix SVD

Can choose $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ orthogonal so $\Sigma = U^T A V = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$:

$$A = U\Sigma V^T \;=\; \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$$A \;=\; \sum_{i=1}^{r} \sigma_i (u_i \circ v_i)$$

where $u_i = U(:,i)$, $v_i = V(:,i)$ and $\mathrm{rank}(A) = r$

# Low Rank Approximations

*Theorem:* If $A = U\Sigma V^T$ is the SVD of $A$ and the singular values are sorted as $\sigma_1 \geq \cdots \geq \sigma_r$, then for any $k < r$, the *best* rank-$k$ approximation to $A$ is

$$B = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

*Low rank approximations are important for data compression*

# Image Compression Example

Mars landscape

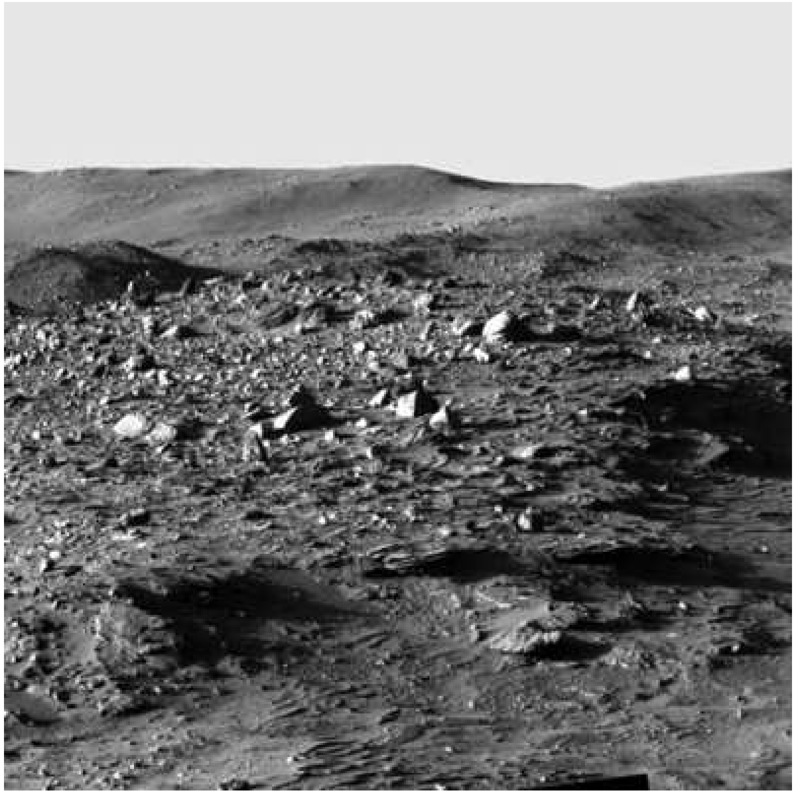
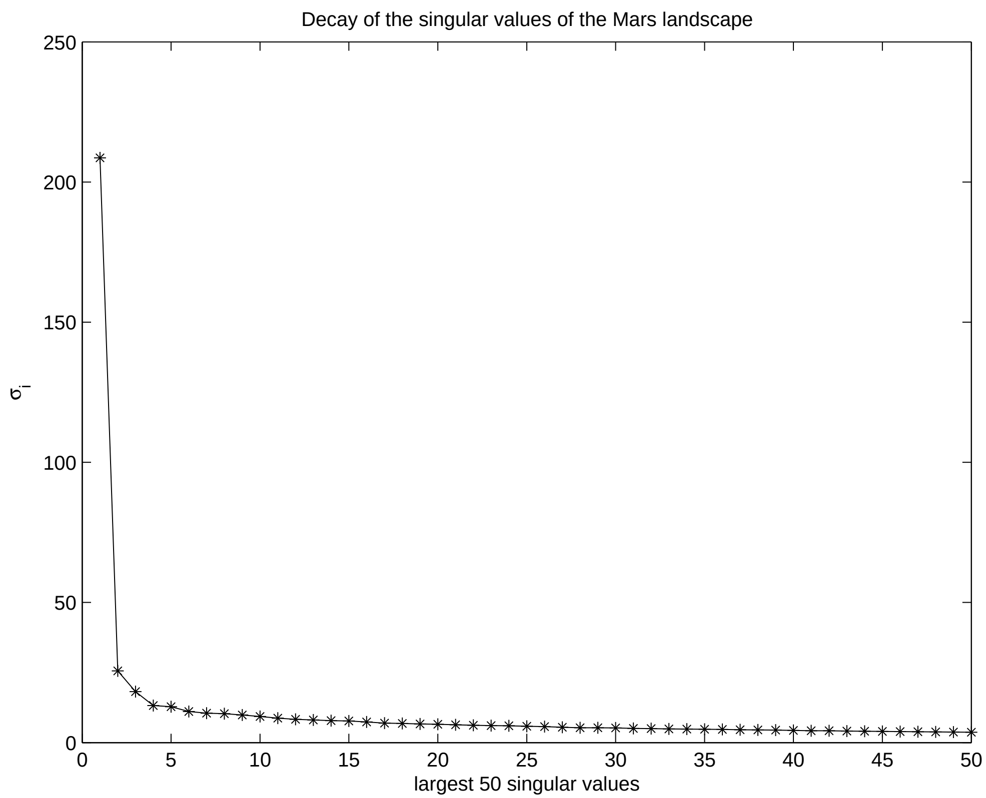


Photo Credit: NASA/JPL-Caltech/Cornell, 2006

A $400 \times 400$ pixel image means sending $160,000$ pixels back to earth!

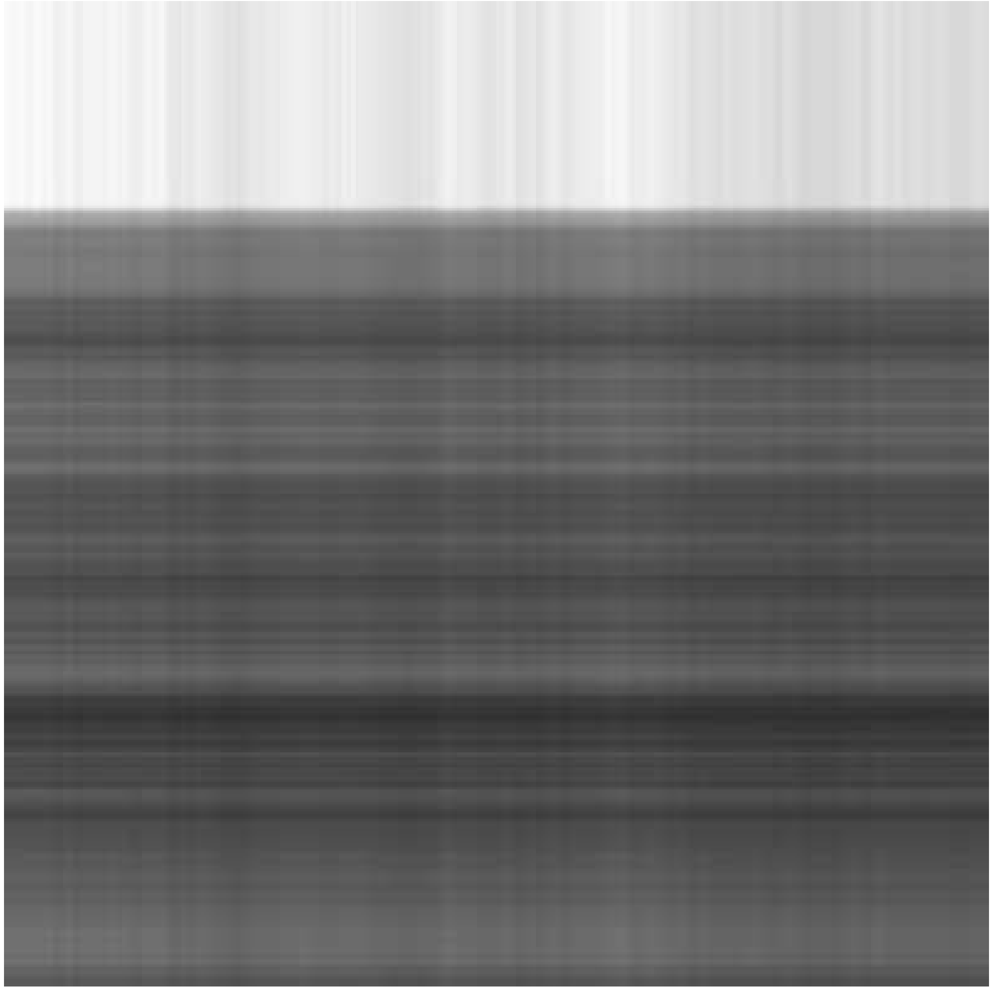
# Singular Values of Mars Image

Stored image $\longrightarrow A \in \mathbb{R}^{400 \times 400}$
$\mathrm{Rank}(A) = 400$



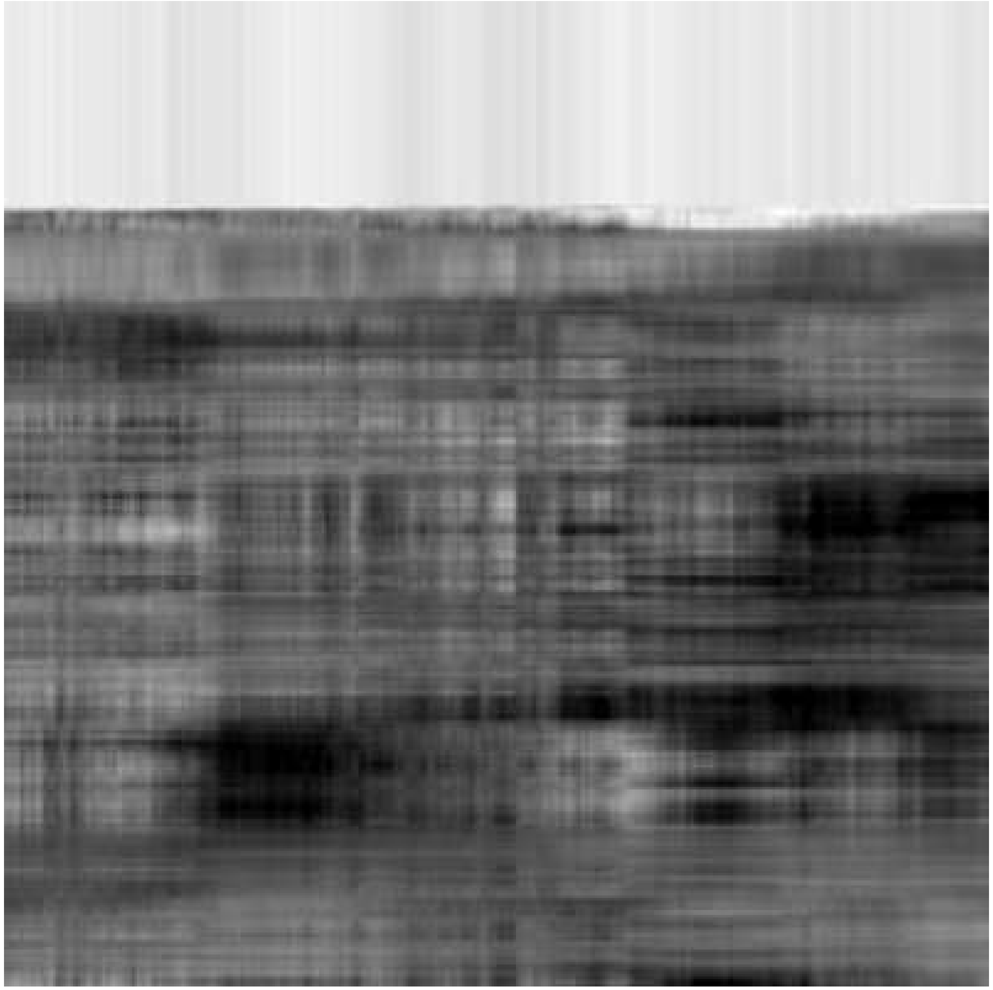

Plot of the Singular Values

# Low Rank Approximations to Mars



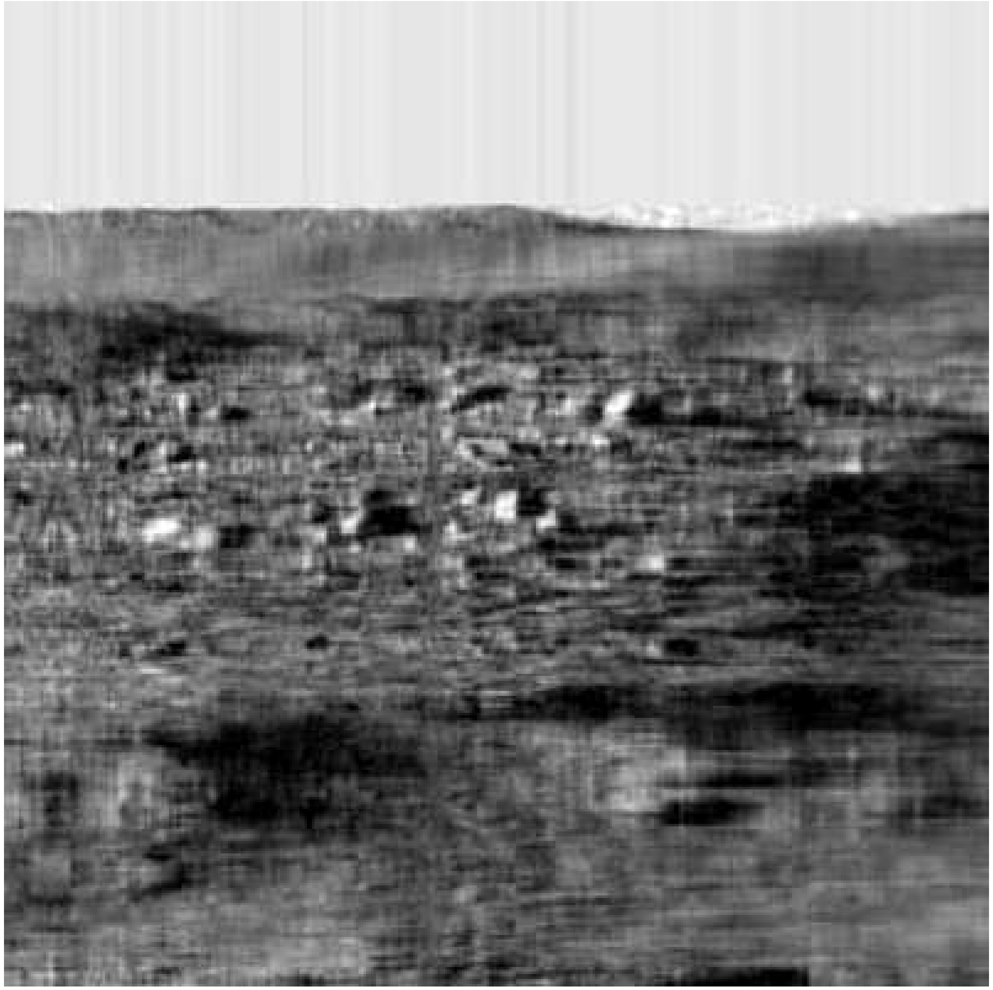
Rank-1 approximation (0.5% of data)

# Low Rank Approximations to Mars



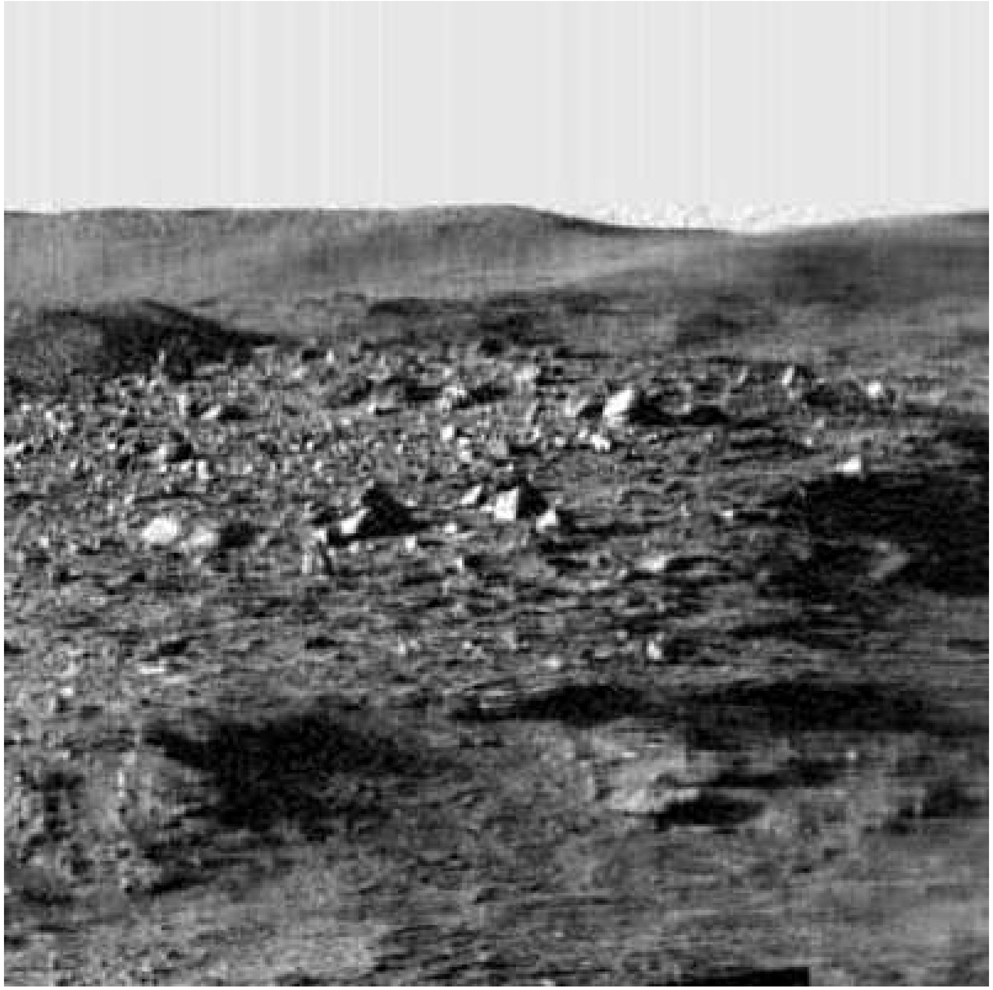
Rank-5 approximation (2.5% of data)

# Low Rank Approximations to Mars



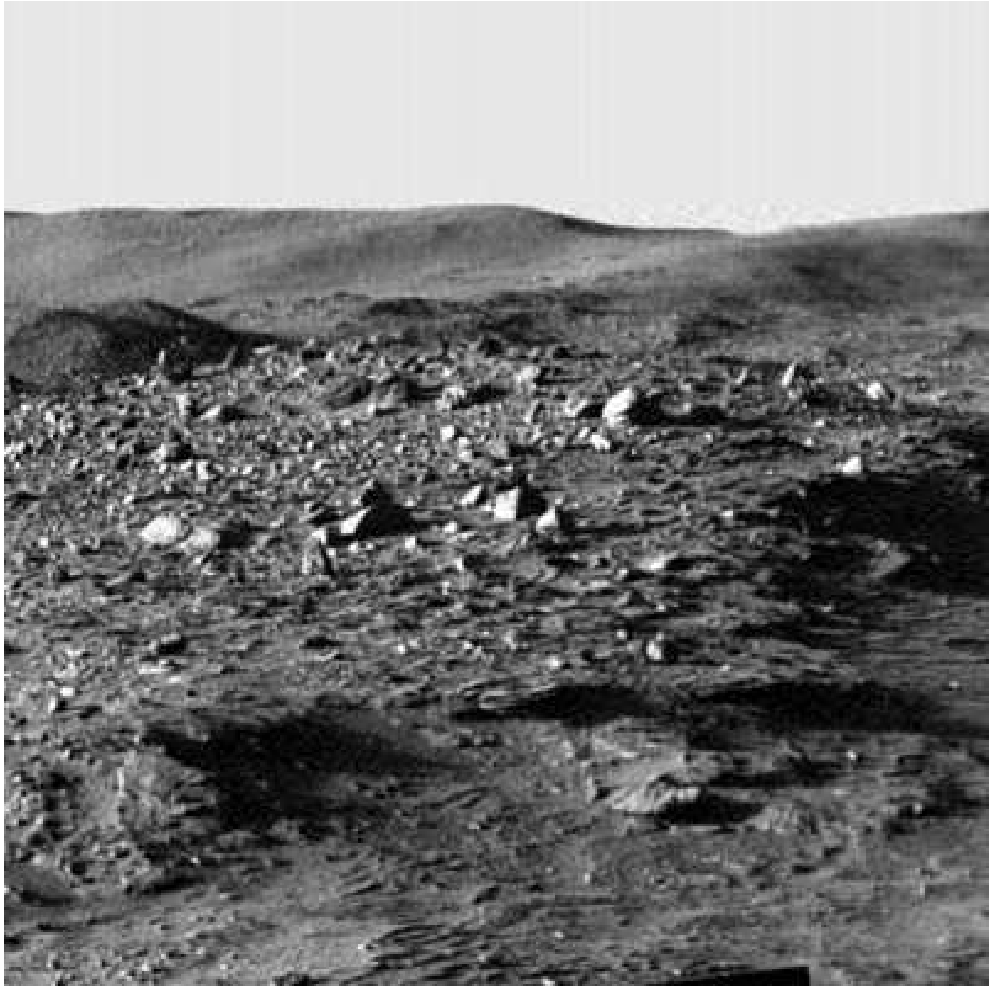
Rank-20 approximation (10% of data)

# Low Rank Approximations to Mars



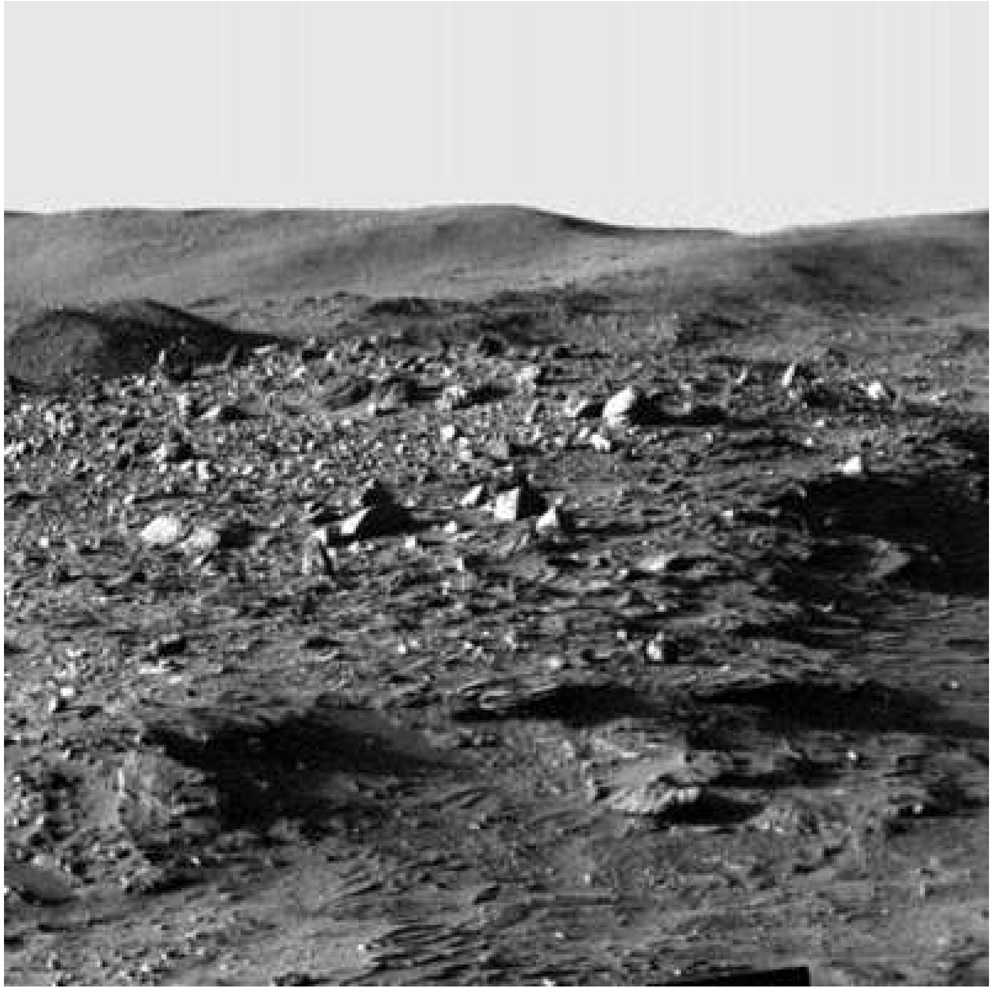
Rank-50 approximation (25% of data)

# Low Rank Approximations to Mars



Rank-100 approximation (50% of data)

# Low Rank Approximations to Mars



Rank-130 approximation (65% of data)

# Low Rank Approximations to Mars

True Image

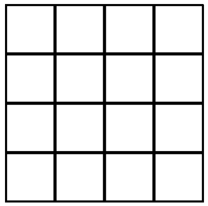
# Applications of the Matrix SVD

- Image Compression
- Noise Filtering
- Handwriting Recognition (used by USPS, and PDAs)
- Statistical Modeling in Chemistry, Psychology, Market Research, and many more areas!

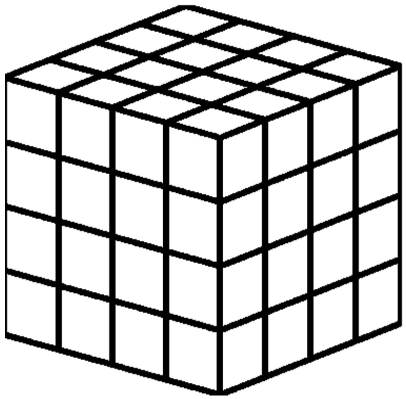
*What about three-dimensional data?*

# What are Tensors?

- Second-order tensor $A = (a_{ij}) \in \mathbb{R}^{n_1 \times n_2}$



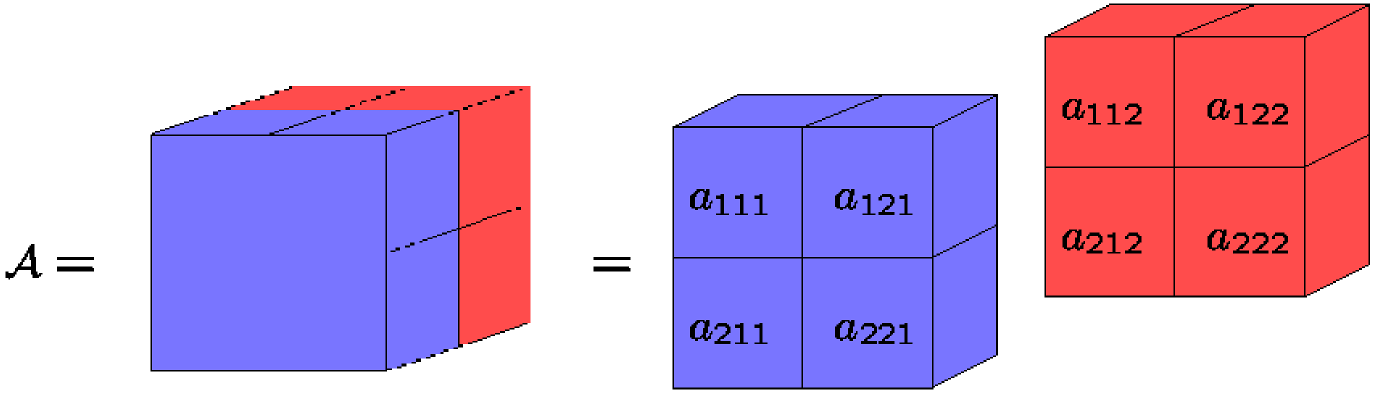
- Third-order tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3}$



- $p^{th}$-order tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_p}) \in \mathbb{R}^{n_1 \times \cdots \times n_p}$

# Representing Tensors

*Example*: $2 \times 2 \times 2$

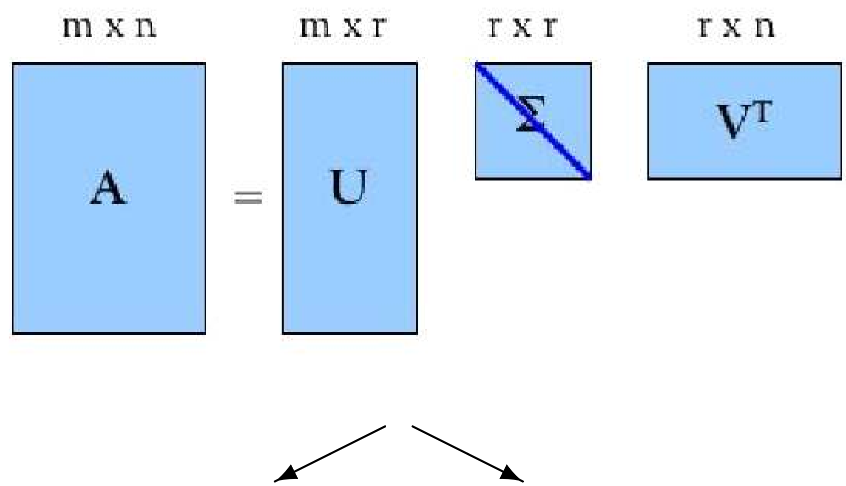



*Rank*:

$$\mathcal{A} = \sum_{i=1}^{r} (u_i \circ v_i \circ w_i)$$

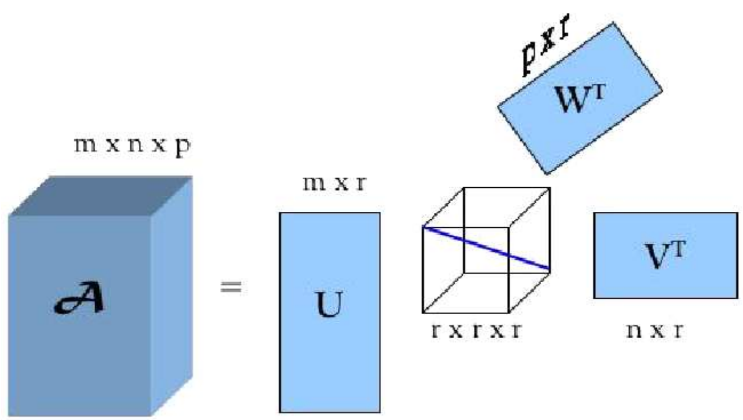
What is the minimum $r$?

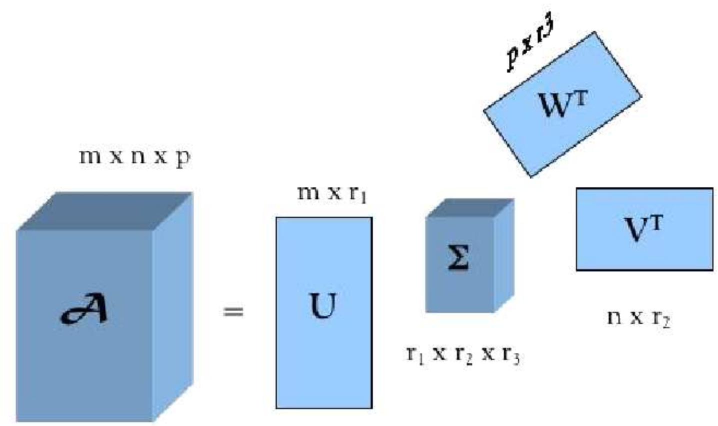
# Orthogonal *or* Diagonal for Tensors




Case 1: Diagonal $\Sigma$

Case 2: Orthogonal $U, V, W$



# Some Facts about Tensor Rank

1. Formula for the maximum possible rank of a tensor of given dimension does not yet exist
2. No known method to compute the “minimum” tensor decomposition directly
3. Minimum tensor representation not necessarily orthogonal

   (Denis and Dhorne, 1989)
4. A tensor over $\mathbb{R}$ may have a different rank then the same tensor considered over $\mathbb{C}$ (Kruskal, 1989)
5. Set of rank-deficient tensors has positive volume

   (Kruskal, 1989)

# Results: Connections between Rank and Generalized Eigenvalues

Given 2 matrices $A, B$, the generalized eigenvalues are given by

$$Ax = \lambda Bx$$

Let $A, B \in \mathbb{R}^{2\times 2}$ (faces of $\mathcal{A}$)

$\boxed{\text{real generalized eigenvalues} \quad \Leftrightarrow \quad \text{maximum rank is 2}}$

$\boxed{\text{complex generalized eigenvalues} \quad \Leftrightarrow \quad \text{rank is 3}}$

$\boxed{\text{repeated generalized eigenvalues} \quad \Leftrightarrow \quad \text{rank is 3}}$

# Results for $n \times n \times 2$ Tensors

If $A, B \in \mathbb{R}^{n\times n}$ and at least one of $A$ and $B$ is invertible with a full set of generalized eigenvectors, then the maximum tensor rank is

$$n + k$$

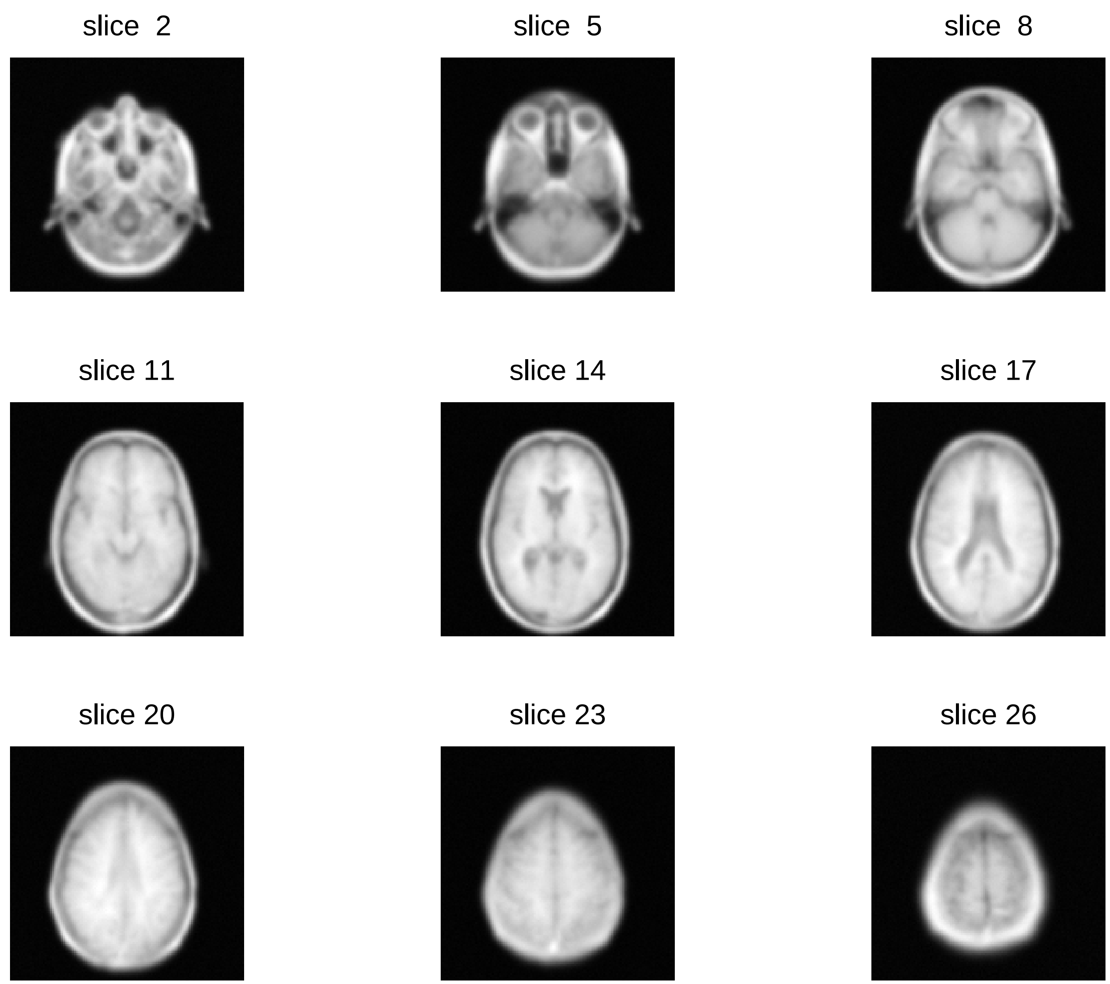
where $k$ is the number of complex conjugate generalized eigenpairs.

*Conclusion*:

$$\text{rank} \;\; \leq \;\; \lfloor 3n/2 \rfloor$$

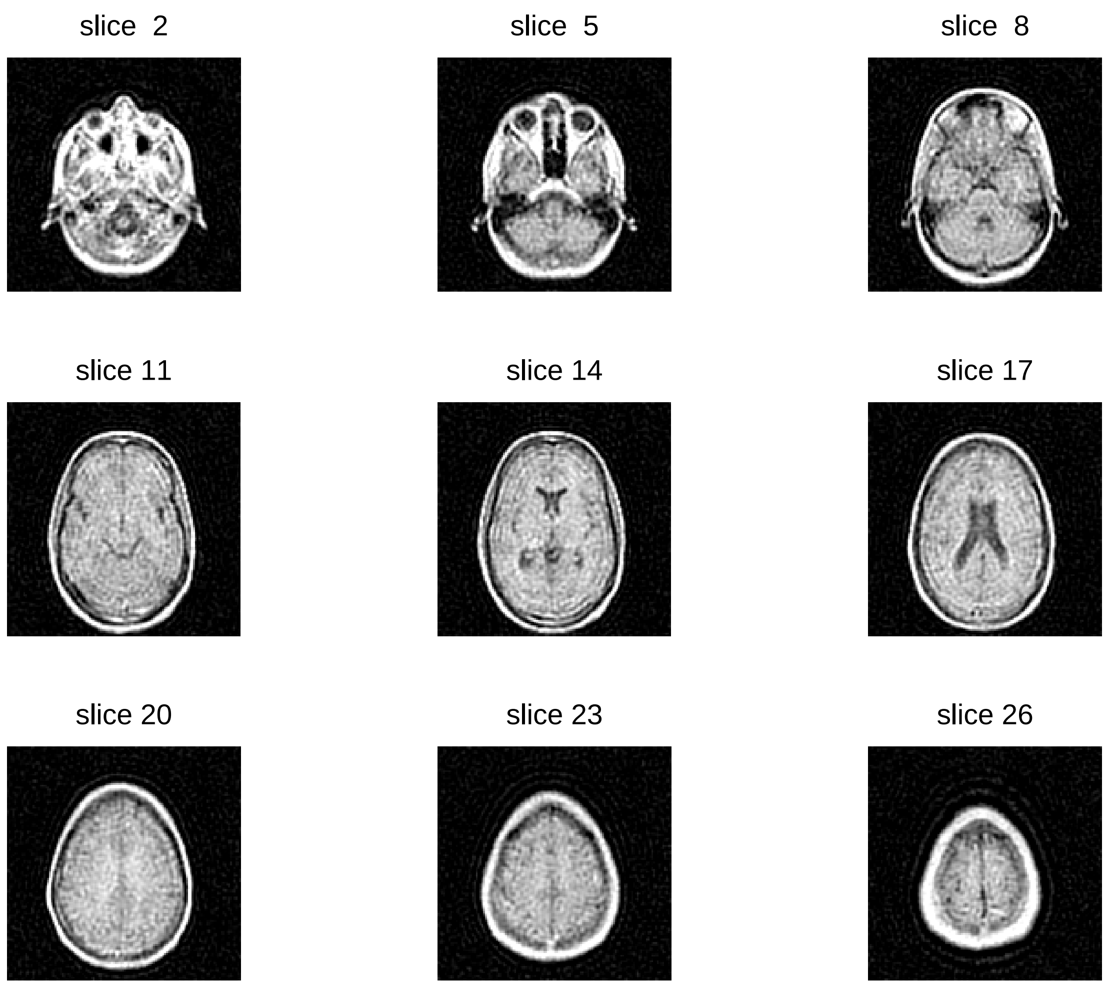
# MRI Scan: Blurred Image (Nagy,Kilmer, 2005)

$128 \times 128 \times 27$ Pixel Image



# MRI Scan: Restored Image

$128 \times 128 \times 27$ Pixel Image



# Conclusions

- Extending rank to higher-order tensors is not a straight-forward extention from the matrix case
- There exists a relationship between tensor rank and linear algebra in simple cases
- Insight into difficulties of the tensor rank problem

Thank you!

carlam@math.jmu.edu