A Higher-order Generalization of the Matrix SVD as a Product of Higher-order Tensors

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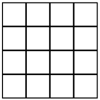
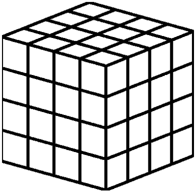
What are Tensors?

- Second-order tensor $A = (a_{ij}) \in \mathbb{R}^{n_1 \times n_2}$



- Third-order tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n_1 \times n_2 \times n_3}$



- p^{th}-order tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_p}) \in \mathbb{R}^{n_1 \times \cdots \times n_p}$

Motivation: A two-way decomposition

Suppose $U \in \mathbb{R}^{M\times M}$, $V \in \mathbb{R}^{N\times N}$ are orthogonal, and $\Sigma = U^T A V$, then

$$A = U\Sigma V^T = \sum_{i=1}^{M}\sum_{j=1}^{N} \sigma_{ij} u_i v_j^T$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{N} \sigma_{ij}(u_i \circ v_j)$$

where $u_i = U(:,i)$, $v_j = V(:,j)$

Tensor Decompositions

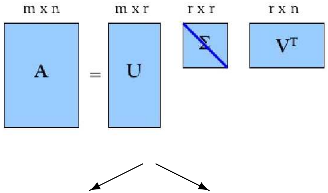
Let $\mathcal{A} \in \mathbb{R}^{M \times N \times P}$

Goal: To find $U \in \mathbb{R}^{M \times M}$, $V \in \mathbb{R}^{N \times N}$, $W \in \mathbb{R}^{P \times P}$, and $\Sigma = (\sigma_{ijk}) \in \mathbb{R}^{M \times N \times P}$ such that

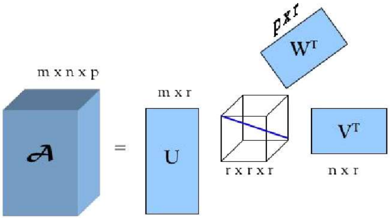
$$\mathcal{A} = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{P} \sigma_{ijk} (u_i \circ v_j \circ w_k)$$

where $u_i = U(:, i)$, $v_j = V(:, j)$, $w_k = W(:, k)$

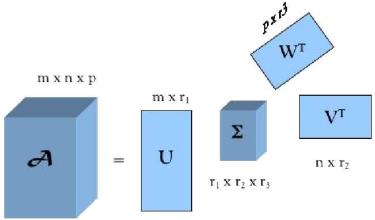
Orthogonal *or* Diagonal for Tensors



Case 1:
Diagonal Σ



Case 2:
Orthogonal U, V, W



Previous Work

Tensor-tensor Multiplication using contracted product...

- Set of all third-order tensors is not closed
- No notion of inverse possible

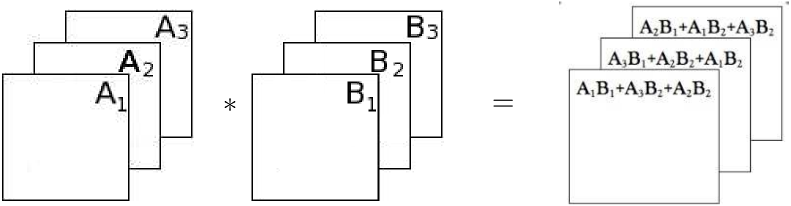
New multiplication operation defined that is closed, inverse exists, that gives way to a new way of thinking about the SVD

New tensor-tensor multiplication

$$\begin{array}{l}\mathcal{A} \in \mathbb{R}^{L \times M \times N} \\ \mathcal{B} \in \mathbb{R}^{M \times P \times N}\end{array} \quad \Rightarrow \quad \mathcal{A} * \mathcal{B} \in \mathbb{R}^{L \times P \times N}$$

- Multiplication defined in terms of the tensor "slices"
- Circulant matrices play a role
- Operation is associative
- Can define an inverse
- Set of $N \times N \times N$ invertible tensors form a group under this multiplication

New tensor-tensor multiplication

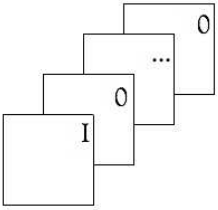


$$\mathcal{A} \quad * \quad \mathcal{B} \quad = \quad \mathcal{C}$$

$$\begin{bmatrix} A_1 & A_3 & A_2 \\ A_2 & A_1 & A_3 \\ A_3 & A_2 & A_1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_3B_2 + A_2B_3 \\ A_2B_1 + A_1B_2 + A_3B_3 \\ A_3B_1 + A_2B_2 + A_1B_3 \end{bmatrix} = \begin{bmatrix} \mathcal{C}(:,:,1) \\ \mathcal{C}(:,:,2) \\ \mathcal{C}(:,:,3) \end{bmatrix}$$

Identity

The $N \times N \times P$ **identity tensor**, $\mathcal{I}$, is the tensor whose frontal face is the $N \times N$ identity matrix and whose other faces are zeros.



In general, $\mathcal{A} * \mathcal{I} = \mathcal{I} * \mathcal{A} = \mathcal{A}$

Inverse

Let $\mathcal{A} \in \mathbb{R}^{N \times N \times N}$. Then the **tensor inverse** of $\mathcal{A}$ is any tensor $\mathcal{B} \in \mathbb{R}^{N \times N \times N}$ such that

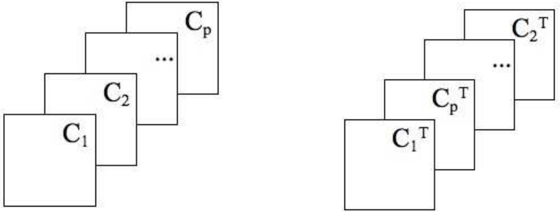
$$\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A} = \mathcal{I}$$

We denote the inverse of $\mathcal{A}$ as $\mathcal{A}^{-1}$.

It follows that $(\mathcal{A} * \mathcal{B})^{-1} = \mathcal{B}^{-1} * \mathcal{A}^{-1}$

Transpose

Let $\mathcal{C} \in \mathbb{R}^{L \times M \times P}$ with faces $C_1, \ldots, C_P \in \mathbb{R}^{L \times M}$. Then



It follows that $(\mathcal{B} * \mathcal{C})^T = \mathcal{C}^T * \mathcal{B}^T$

Frobenius Norm and Orthogonality

Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{L \times M \times N}$. Then the **Frobenius norm** of $\mathcal{A}$ is

$$||\mathcal{A}||_F = \sqrt{\sum_{i=1}^{L} \sum_{j=1}^{M} \sum_{k=1}^{N} a_{ijk}^2}$$

Let $\mathcal{Q} \in \mathbb{R}^{N \times N \times P}$. $\mathcal{Q}$ is **orthogonal** if

$$\mathcal{Q}^T * \mathcal{Q} = \mathcal{Q} * \mathcal{Q}^T = \mathcal{I}$$

If $\mathcal{A}$ is a tensor, then it follows that

$$||\mathcal{Q} * \mathcal{A}||_F = ||\mathcal{A}||_F$$

Tensor SVD

Let $\mathcal{A} \in \mathbb{R}^{L \times M \times N}$. Then $\mathcal{A}$ can be factored as

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$$

where $\mathcal{U} \in \mathbb{R}^{L \times L \times N}$ and $\mathcal{V} \in \mathbb{R}^{M \times M \times N}$ are orthogonal tensors and $\mathcal{S} \in \mathbb{R}^{L \times M \times N}$ has diagonal matrix faces.

If $\mathcal{A} \in \mathbb{R}^{N \times N \times N}$,

$$\mathcal{A} = \sum_{i=1}^{N} \mathcal{U}(:, i, :) * \mathcal{S}(i, i, :) * \mathcal{V}(:, i, :)^T$$

Tensor SVD: computation

- Computation of the tensor SVD involves SVDs of block diagonal elements obtained from block diagonalizing the circulant matrix generated by $\mathcal{A}$
- Using the SVDs of the blocks leads to algorithms for compression
- Operation extends recursively to order-p tensors when $p > 3$

One Compression Strategy

Suppose $\mathcal{A} \in \mathbb{R}^{N \times N \times N}$

Can prove that if $\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$ then

$$\sum_{i=1}^{N} \mathcal{U}(:,:,i) \qquad \text{is orthogonal.}$$

Therefore

$$\sum_{i=1}^{N} \mathcal{A}(:,:,i) = \left(\sum_{i=1}^{N} \mathcal{U}(:,:,i) \right) \left(\sum_{i=1}^{N} \mathcal{S}(:,:,i) \right) \left(\sum_{i=1}^{N} \mathcal{V}(:,:,i) \right)^T$$

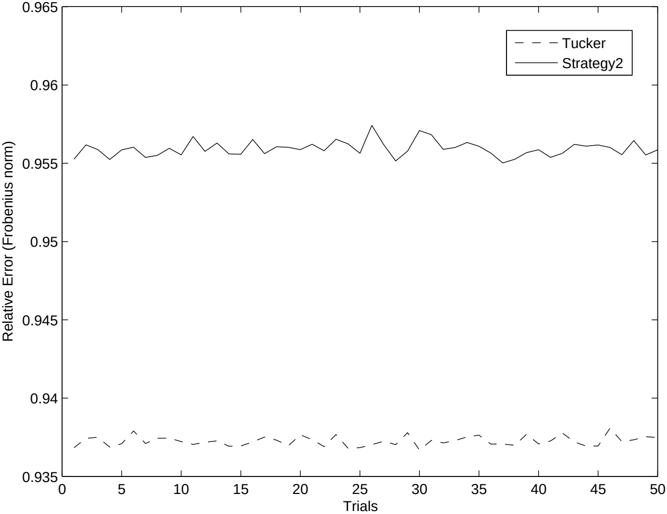
One Compression Strategy

$$\sum_{i=1}^{N} \mathcal{A}(:,:,i) = \left(\sum_{i=1}^{N} \mathcal{U}(:,:,i)\right) \left(\sum_{i=1}^{N} \mathcal{S}(:,:,i)\right) \left(\sum_{i=1}^{N} \mathcal{V}(:,:,i)\right)^{T}$$

- Take rank-k approximation and "build" back approximation to the tensor $\mathcal{A}$
- Can rewrite "compressed" tensor $\mathcal{A}_c$ as sum of outer products:

$$\mathcal{A} \approx \mathcal{A}_c = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \tilde{U}(:,i) \circ \tilde{V}(:,j) \circ \mathcal{M}(i,j,:)$$

Numerical Results



Summary

- New definitions allow for greater flexibility in design of compression algorithms for tensors
- Can be extended to other matrix factorizations (e.g., QR)
- Computationally efficient
- Allows for weighting of tensor faces

Thank you!