Journal of Pure and Applied Algebra 70 (1991) 251–261

Tree Groups and the 4 String Pure Braid Group

Carl Droms *
James Madison University
Harrisonburg, VA 22807

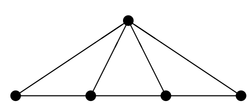
Jacques Lewin
Syracuse University
Syracuse, NY 13244-1154

Herman Servatius
College of the Holy Cross
Worcester, MA 01610

Abstract

Given a graph Γ, undirected, with no loops or multiple edges, we define the *graph group* on Γ, F_Γ, as the group generated by the vertices of Γ, with one relation $xy = yx$ for each pair x and y of adjacent vertices of Γ.

In this paper we will show that the unpermuted braid group on four strings is an HNN-extension of the graph group F_S, where

$S =$ 

The form of the extension will resolve a conjecture of Tits for the 4-string braid group. We will conclude, by analyzing the subgroup structure of graph groups in the case of trees, that for any tree T on a countable vertex set, F_T is a subgroup of the 4-string braid group.

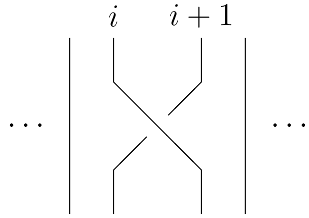
We will also show that this uncountable collection of subgroups of the 4-string braid group is linear, that is, each subgroup embeds in GL(3,$\mathbb{R}$), as well as embedding in Aut(F), where F is the free group of rank 2.

1 Tree Groups and Braid Groups

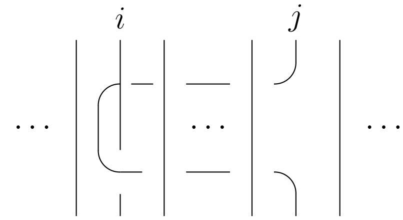
In this section we are concerned only with the four string braid group, B_4,

*This author was supported by a summer research grant from James Madison University

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle.$$



σ_i



$A_{ij} = \sigma_{j-1} \dots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$

See [3] for background on the braid group. The main result here is that the graph group on embeds naturally into B_4.

There is a map $B_4 \to S_4$, into the symmetric group which associates to each braid the permutation of the four strings it effects. The kernel of this map is $P = P_4$ the *pure* or *unpermuted* braid group. P has a presentation, see [10] pg 174, as follows:

GENERATORS: A_{12}, A_{23}, A_{34}, A_{13}, A_{24}, A_{14}

RELATIONS:

$$\begin{aligned}
A_{12}A_{34} &= A_{34}A_{12} \\
A_{14}A_{23} &= A_{23}A_{14} \\
A_{24}A_{13}A_{24}^{-1} &= A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{13}A_{12}^{-1}A_{14}^{-1}A_{12}A_{14} \\
A_{23}A_{12}A_{23}^{-1} &= A_{13}^{-1}A_{12}A_{13} \\
A_{24}A_{12}A_{24}^{-1} &= A_{14}^{-1}A_{12}A_{14} \\
A_{34}A_{13}A_{34}^{-1} &= A_{14}^{-1}A_{13}A_{14} \\
A_{34}A_{23}A_{34}^{-1} &= A_{24}^{-1}A_{23}A_{24} \\
A_{23}A_{13}A_{23}^{-1} &= A_{13}^{-1}A_{12}^{-1}A_{13}A_{12}A_{13} \\
A_{24}A_{14}A_{24}^{-1} &= A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{14} \\
A_{34}A_{14}A_{34}^{-1} &= A_{14}^{-1}A_{13}^{-1}A_{14}A_{13}A_{14} \\
A_{34}A_{24}A_{34}^{-1} &= A_{24}^{-1}A_{23}^{-1}A_{24}A_{23}A_{24}
\end{aligned}$$

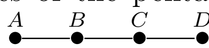
The subgroup of P consisting of all braids which are concentrated on strings i though j has, [9], an infinite cyclic center generated by $D_{ij} = (\sigma_i\sigma_{i+1} \dots \sigma_{j-1})^{j-i+1}$. The element D_{ij} geometrically represents a full twist of the i'th through j'th strings through a full 360 degrees.

There is thus a map f from the graph group on the graph into P, $f : F_\Gamma \to P$, defined by $f(A) = D_{12}$, $f(B) = D_{13}$, $f(C) = D_{23}$, $f(D) = D_{24}$, $f(E) = D_{34}$, and $f(X) = D_{14}$. The relations of F_Γ are preserved under f, with $D_{12}D_{34} = D_{34}D_{12}$ since D_{12} and D_{34} are concentrated on disjoint sets of strings. f is also surjective since we have the equations

$$A_{12} = D_{12} \quad A_{13} = D_{12}^{-1}D_{13}D_{23}^{-1} \quad A_{14} = D_{14}D_{13}^{-1}D_{23}D_{24}^{-1}$$

$$A_{23} = D_{23} \quad A_{24} = D_{23}^{-1} D_{24} D_{34}^{-1}$$
$$A_{34} = D_{34}$$

It is straightforward to verify that the kernel of f is the normal closure of the element $AB^{-1}CD^{-1}EA^{-1}BC^{-1}DE^{-1}$. Setting Γ' equal to the outer pentagon of Γ, it follows that P is isomorphic to $\langle X \rangle \oplus G$, where G is the group obtained from $F_{\Gamma'}$ by adjoining the single relation $AB^{-1}CD^{-1}E = ED^{-1}CB^{-1}A$.

Any four consecutive vertices of the pentagon generate a three line group, say, $\{A, B, C, D\}$. Setting $\Omega =$ 

we have homomorphisms $F_\Omega \to F_{\Gamma'} \to G$, where the first map is simply the inclusion, and we want to show this composit is injective. This follows from

THEOREM 1 *G is an HNN extension of F_Ω.*

PROOF: Consider the set $\{A, D, AB^{-1}CD^{-1}\}$. These three elements generate a free subgroup of rank three in F_Ω. This is because the so-called "edge group", $\langle AB^{-1}, B^{-1}C, C^{-1}D \rangle$ is free, [6], and the automorphism of F_Ω induced by

$$A \to AB; B \to B; C \to C; D \to CD$$

carries this subgroup onto $\langle A, B^{-1}C, D \rangle$ which equals $\langle A, AB^{-1}CD^{-1}, D \rangle$ by Nielsen transformations. Similarly $\langle A, D^{-1}CB^{-1}A, D \rangle$ is free of rank 3, and so there is an isomorphism

$$f : \langle A, D, D^{-1}CB^{-1}A \rangle \to \langle A, AB^{-1}CD^{-1}, D \rangle$$

given by $f(A) = A$, $f(D) = D$, and $f(D^{-1}CB^{-1}A) = f(AB^{-1}CD^{-1})$. Let H be the HNN extension

$$\langle F_\Omega, E \mid g^E = f(g), g \in \langle A, D, AB^{-1}CD^{-1} \rangle \rangle.$$

By examining presentations, we see that H is isomorphic to G. $\square$

Note that the other four three line groups also inject by symmetry.

COROLLARY 1 *F_Ω embeds into P.*

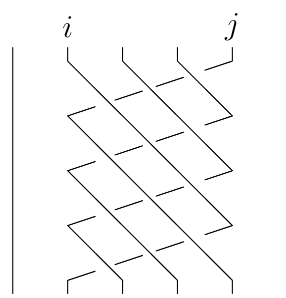
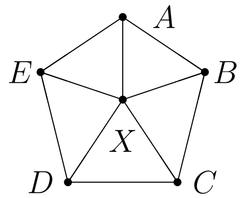


Figure 1: The element D_{ij}

$\Gamma =$ 

Corollary 1, together with Corollary 2 of Theorem 3 give

Corollary 2 *The pure 4-string braid group has uncountably many non- isomorphic subgroups.*

The following corollary answers in the affirmative, in the case of the 4-string braid group, a conjecture due to Tits, [1].

Corollary 3 *The subgroup of B_4 generated by D_{12}, D_{23} and D_{34} has the presentation $\langle D_{12}, D_{23,}D_{34} \mid [D_{12}, D_{34}]\rangle$.*

Proof: The subgroup of B_4 generated by D_{12}, D_{13}, D_{23} and D_{34} has the presentation

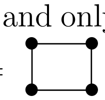
$$\langle D_{12}, D_{13}, D_{23}, D_{34} \mid [D_{12}, D_{13}], [D_{13}, D_{23}], [D_{12}, D_{34}]\rangle,$$

and we have already noted, [8], that every subset of the vertices of a graph group generates a subgroup which is itself a graph group on their induced subgraph. □

Corollary 4 *Let P denote the pure 4-string braid group, $P = \langle X\rangle \oplus G$ as above. The presentation for G is Cohen-Lyndon aspherical.*

Proof: This follows from Theorems 4.3 and 3.7 of [4]. □

2 Subgroups of Graph Groups

If Γ is finite, then [8] every f.g. subgroup of F_Γ is a itself a graph group if and only if no full subgraph of Γ is isomorphic to either $\Omega =$ or $C_4 =$ . The graph group on C_4 is the direct product of two free groups of rank two, whose subgroup structure was examined in [2]. In this section, we consider the related question: If Γ is a graph, for which graphs Φ does F_Γ have a subgroup isomorphic to F_Φ? We will answer this question in the case that Γ is a tree.

If F_Φ is isomorphic to a subgroup of F_Γ, then we say F_Φ is a *graph subgroup* of F_Γ. In particular, if Φ is a full subgraph of Γ, that is, every pair of vertices in Φ which are adjacent in Γ are also adjacent in Φ, then the subgroup of F_Γ generated by the vertices of Φ is isomorphic to F_Φ. In this case we call F_Φ a *subgraph subgroup* of F_Γ. Henceforth, we will use the term "subgraph" to mean "full subgraph".

We will show that, for any subgraph Φ of Γ, the normal closure of the subgraph subgroup F_Φ in F_Γ is itself a graph group. Indeed, one can describe quite simply a set of "graphic" generators and relations for this subgroup. We denote by Φ' the subgraph of Γ generated by the vertices of Γ which are not in Φ. In this section we will prove

THEOREM 2 *The normal closure of F_Φ in F_Γ has a presentation with generators g', one for each distinct nontrivial conjugate $g = a^{-1}\mu a \in F_\Gamma$, where $a \in F_{\Phi'}$ and μ is a vertex of Φ, and one relator $[g', h']$ for each such pair g, h that commutes in F_Γ.*

PROOF: For any graph Λ, let S_Λ be the standard complex for F_Λ, so that Λ has one 0-cell, 1, the base point, a 1-cell v for each vertex v of Λ, and a 2-cell v_1——v_2 for each edge of Λ. The 2-cell v_1——v_2 is attached to the path $v_1 v_2 v_1^{-1} v_2^{-1}$. By construction, $\pi_1(S_\Lambda) = \pi_1(S_\Lambda, 1) = F_\Lambda$.

In the notation of the theorem, there is a retraction $f : S_\Gamma \to S_{\Phi'}$ which collapses the 1-cells corresponding to vertices of Φ to 1. If v_1——v_2 is an edge of Φ, v_1——v_2 is collapsed to 1. If v_1——v_2 has $v_1 \in \Phi$, $v_1 \in \Phi'$, then the cell v_1——v_2 is collapsed to the 1-cell v_2 of Φ'. The kernel $K = \ker f_*$ of the induced map $f_* : F_\Gamma \longrightarrow F_{\Phi'}$ is clearly the normal closure of the subgroup F_Φ in F_Γ

The universal cover $U_{\Phi'}$ of $S_{\Phi'}$ has in induced cell structure as a 2-dimensional complex and the 1-skeleton is the Cayley graph of $F_{\Phi'}$. Consider the pullback diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f'} & U_{\Phi'} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S_\Gamma & \xrightarrow[f]{} & S_{\Phi'} \end{array}$$

So that Y is the subspace of $S_\Gamma \times U_{\Phi'}$ consisting of all points (x, y) with $f(x) = p(y)$. Since the pullback of a covering map is a covering map, the induced map p'_* in

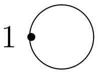
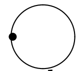
$$\begin{array}{ccc} \pi_1(Y) & \longrightarrow & 1 \\ {\scriptstyle p'_*}\downarrow & & \downarrow \\ F_\Gamma & \xrightarrow[f_*]{} & F_{\Phi'} \end{array}$$

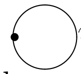
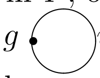
is injective and maps into K. The loops $v_1 v v_1^{-1}$ with $v \in \Phi$, $v_1 \in \Phi'$ of S_Φ are easily seen to lift to loops of Y, so p'_* gives an isomorphism $\pi_1(Y) \cong K$.

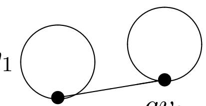
It is straightforward to give a convenient decomposition of Y as a 2-complex.

First, by considering $S_{\Phi'}$ a a subcomplex of S_Γ, and identifying $U_{\Phi'}$ with the graph of p, we see that we can take $U_{\Phi'}$ to be a subcomplex of Y.

We take $F_{\Phi'}$, the 0-skeleton of $U_{\Phi'}$, to be the 0-skeleton of Y.

The 1-cells of $U_{\Phi'}$ are edges g——gv, $v \in \Phi$, $g \in F_\Phi$. The loops 1  v, $v \in \Phi$ of S_Γ pull back to loops g  v, $g \in F_\Gamma$.

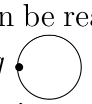
Aside from the 2-cells of $U_{\Phi'}$, we have 2-cells which are pullbacks of 2-cells of $S - \Gamma$. If v_1——v_2 is a 2-cell with both $v_1, v_2 \in \Phi$, it pulls back to torrii in Y, one for each $g \in F_{\Phi'}$, whose meridians and longitudes are g  v_1 and g  v_2. If $v_1 \in \Phi$ and $v_2 \in \Phi'$, then v_1——v_2 pulls back to a cylinder attached along:



It is easily checked that every point of Y lies on one of the cells described above.

We now conider the quotient complex $\overline{Y}$ which is obtained from Y by identifying the subcomplex $U_{\Phi'}$ to a point. Since $U_{\Phi'}$ is connected and simply connected, the exact homotopy sequence of a pair

$$\cdots \to \pi_1(A,1) \to \pi_1(Y,1) \to \pi_1(Y,A) \to \pi_0(A,1) \to \cdots$$

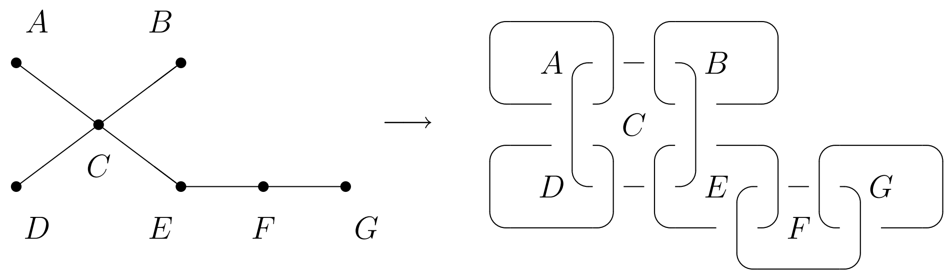
implies that the quotient map $Y \longrightarrow \overline{Y}$ induces an isomorphism $\pi_1(Y,1) \longrightarrow \pi_1(\overline{Y},\overline{1}) = \pi_1(Y,A)$. $\overline{Y}$ is a 2-complex with one 0-cell, the class of 1, so it is a standard complex, and a presentation for $\pi_1(Y,1) \cong \pi_1(\overline{Y},\overline{1})$ can be read off from the induced cell structure of $\overline{Y}$. If we write v^g for the 1-cells g  v, then we have as generators for K the set $\{v^g \mid v \in \Phi, g \in F_{\Phi'}\}$. The relations are

1. $v^g = v^{gw}$ if v——w is an edge of Γ with $v \in \Phi$ and $w \in \Phi'$.
2. $[v^g, w^g] = 1$ is v——w is an edge of Φ.

The centralizer $C_{\Phi'}(v)$ of v in $F_{\Phi'}$ is generated by the elements of ϕ' adjacent to v, as is easily seen from the normal form of elements in F_Γ (or see [11]). The relations in (i) can then be eliminated if we choose for generators of K elements v^g in 1:1 correpondence with the cosets of $C_{\Phi'}$.

Since the element v^g is in fact the conjugate gvg^{-1}, the theorem is proved. □

We now consider trees. It has been shown [6] that if T is a finite tree, then F_T is a three-manifold group. It can be shown, in fact, that for any locally finite tree T, F_T is the group of a link in $\mathbb{R}^3$. The construction of the link in the case of a finite tree is illustrated in the diagram below
So to each vertex in the tree there corresponds one unknotted circle, linked once to each of its neighbors as shown. A van Kampen argument shows that the fundamental group of the complement of this link is F_T, and by [7] it follows that two distinct trees give two non-equivalent links. Nevertheless, any two finite trees with the same number of vertices yield links with identical Jones polynomials, as can easily be verified.



We will now characterize the graph subgroups of tree groups. A tree T is called a star if it has one vertex which is adjacent to all the others. Note that this is the case if and only if T has no subgraph isomorphic to L_3, and that if T is a star, then F_T is the direct product of an infinite cyclic group and a free group.

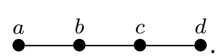
Proposition 1 *If T is a star, then every subgroup of F_T is either free or isomorphic to F_S for some star S.*

Proof: Since T is a star, F_T is isomorphic to $\mathbb{Z} \oplus F$, where F is free. Thus, if $H < F_T$, then H is a semidirect product of $H \cap \mathbb{Z}$ and the projection of H onto the factor F, and since $\mathbb{Z}$ lies in the center of F_T, the product is direct. Thus, H isomorphic to $(H \cap \mathbb{Z}) \oplus K$, where K is free. Since $H \cap \mathbb{Z}$ is either trivial or infinite cyclic, the result follows. □

Theorem 3 *If T is a tree which is not a star, then every graph subgroup of T is isomorphic to F_Φ for some countable forest Φ. Conversely, if Φ is any countable forest, then F_T has a subgroup isomorphic to F_Φ.*

Proof: Let F_Φ be a subgroup of F_T. By [6], F_T is coherent. Thus F_Φ is coherent, which, by [6], implies that every circuit of Φ has a chord. To show that Φ must be a forest, it will suffice to show that Φ may contain no triangles. Suppose Φ contains a triangle Σ. Then there is a finite subgraph T' of T such that $F_\Sigma \langle F_{T'}$. But T' is a forest, so by [5], $cd(F_{T'}) \leq 2$, which is a contradiction, since F_Σ is free abelian of rank 3.

It suffices to prove the second statement for the graph

$\Omega =$  .

It follows from Theorem 2 that the normal closure of the elements b, c in F_Ω is generated by the distinct conjugates of b and c, with defining relators those commutators among pairs of these conjugates which are trivial in F_Ω. The corresponding graph Φ is a forest, by the above, and in fact each vertex of Φ has infinitely many neighbors; consider the element $g^{-1}bg$, where $g \in \mathrm{gp}\langle a, d\rangle$. The elements $(g^{-1}a^{-n})c(a^n g)$ are distinct for distinct $n \in \mathbb{Z}$, and each commutes with $g^{-1}bg$, so the vertex in Φ corresponding to $g^{-1}bg$ has infinitely many neighbors.

Now, it is clear that if Σ is any countable forest, then the graph Φ contains an isomorphic copy of Σ as a full subgraph, and hence F_Ω contains a subgroup isomorphic to F_Σ. □

Corollary 5 *The graph subgroups of F_Ω consist of the graph groups on countable forests.*

The following corollary is analogous to a result in [2] for the direct product of two free groups of rank 2.

Corollary 6 *If the graph Γ contains a full subgraph isomorphic to Ω, then F_Γ has uncountably many non-isomorphic subgroups.*

Proof: This follows immediately from the fact that if the graphs Γ and Φ are not isomorphic, then neither are the groups F_Γ and F_Φ [7], and that there are uncountably many countable trees. $\Box$

3 Representations of Tree Groups*

The question of the linearity of the braid groups is has a long history. The purpose of this section is to prove

Theorem 4 *The group F_Ω has a faithful, 3×3, real representation,* $\Omega = \overset{A}{\bullet}\!\!-\!\!\!-\!\!\overset{B}{\bullet}\!\!-\!\!\!-\!\!\overset{C}{\bullet}\!\!-\!\!\!-\!\!\overset{D}{\bullet}.$

Proof: Let g be the group with presentation

$$\langle A, B, T \mid [A, B] = 1, [B, B^T] = 1, T^2 = 1\rangle$$

Then, applying the Reidemeister-Schrier process, the normal closure of the elements $A, B \in G$ is isomorphic to F_Ω under the correspondence

$$A \to A, B \to B, C \to B^T, D \to A^T.$$

Thus, it will suffice to show that G has such a representation. Let

$$\begin{aligned} G_1 &= \langle A, B \mid [A, B] = 1\rangle \\ G_2 &= \langle B, T \mid [B, B^T] = 1, T^2 = 1\rangle \end{aligned}$$

Then G is isomorphic to the amalgamated free product $G_1 \star_{\langle B\rangle} G_2$.

We will first write faithful representations of G_1 and G_2:

G_1 : G_1 is faithfully represented by the group M_1 generated by the matrices A and B, where

$$A = \begin{pmatrix} y & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

x, y and z are algebraically independent transcendentals.

0* The authors acknowledge with thanks the suggestions of A. Schofield.

G_2: The normal closure in G_2 of the element B is a free abelian group generated by B and B^T. Thus, G_2 is the split extension of this subgroup by the group of order two generated by T. Let T and B be the matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix}$$

Then B^T and B indeed generate a free abelian group of rank 2, so the group M_2 generated by B and T is isomorphic to G_2 by the 5-lemma.

Note that these representations are over the field $\mathbb{Q}(x, y, z)$.

Let U be the space spanned by the column vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and W that spanned by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then $A^nB^m = \begin{pmatrix} y^nx^m & 0 & 0 \\ 0 & z^nx^m & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the only such matrices which act as scalars on U mod W are the powers of B (ie, those with $n = 0$).

In M_2, every element can be written in one of the two forms B^nC^m or TB^nC^m, where $C = B^T$.

$$B^nC^m = \begin{pmatrix} x^{n+m} & 0 & 0 \\ 0 & x^n & 0 \\ 0 & 0 & x^m \end{pmatrix} \quad \textit{and} \quad TB^nC^m = \begin{pmatrix} x^{n+m} & 0 & 0 \\ 0 & 0 & x^m \\ 0 & x^n & 0 \end{pmatrix}$$

and so the only matrices in M_2 that act as scalars on U mod W are again the powers of B. thus, by Wehrfritz's Theorem 3 in [12], the group $M_1 \star_{\langle B \rangle} M_2$, which is isomorphic to G, has a faithful 3-dimensional representation over a purely transcendental extension of the field $\mathbb{Q}(x, y, z)$, and hence over $\mathbb{R}$. □

We define now the group $G = \langle a, b \mid [a^b, a] \rangle$. G contains the infinite line group as the normal closure of a,

$$\cdots \text{ — } b^2ab^{-2} \text{ —— } bab^{-1} \text{ —— } a \text{ —— } b^{-1}ab \text{ —— } b^{-2}ab^2 \text{ — } \cdots,$$

again by the Reidemeister-Schrier process, and hence G contains a copy of the three line group and hence every forest group. It is also true that G has a faithful representation in 4×4 matrices over R as follows. Let X, T and T' be defined by

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \quad T = \begin{pmatrix} t_1 & t_2 \\ t_4 & t_3 \end{pmatrix} \quad T' = \begin{pmatrix} t_1' & t_2' \\ t_4' & t_3' \end{pmatrix},$$

where the non-zero entries are algebraicly independent trancendentals.

Let π be the 4×4 matrix corresponding to the transposition (23).

Then a is represented by $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and b by $\pi \begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix}$ Thus every line group has a 4×4 representation in which the generators are conjugates.

We may also represent tree groups in the automorphism group of a free group.

Theorem 5 *F_Ω, and hence every contable forest group, is contained in the group of automorphisms of a free group of rank 2.*

The group $G = \langle a, b \mid [a^b, a]\rangle$, defined above, also has the presentation

$$G = \langle x, y, t \mid x^t = xy, y^t = y\rangle,$$

under the correspondence $t = a$, $x = b$, and $y = [b, a]$. Note that the subgroup generated by x and y is normal, free of rank two, and the quotient is infinite cyclic, in fact, it is the normal closure of b. Thus, G fits into a split exact sequence

$$1 \to F_2 \to G \to Z \to 1,$$

where $F_2 = \mathrm{gp}\langle x, y\rangle$.

Now, as usual, to this extension corresponds an automorphism s of F_2; in this case, it's the automorphism

$$s(x) = xy \qquad s(y) = y.$$

It is not hard to see that G is actually the subgroup of $Aut(F_2)$ generated by the inner automorphisms and the automorphism s. Thus, every forest group is contained in $Aut(F_2)$.

References

[1] K. Appel and P. Schupp, Artin groups and infinite coxeter groups, Invent. Math. 72 (1983), 201-220.

[2] G. Baumslag and J. Roseblade, Subgroups of direct products of free groups, J. London Math. Soc. (2), 30 (1984), 44-52.

[3] J. Birman, Braids Links and Mapping Class Groups, Princeton U. Press, 1974.

[4] I.M. Chiswell, D.J. Collins, J. Huebschmann, Aspherical group presentations, Math. Z. 178 (1981) 1-36.

[5] W. Dicks, An exact sequence for rings of polynomials in partly commuting indeterminates, J. Pure Appl. Algebra 22 (1981), 215-228.

[6] C. Droms, Graph groups, coherence, and three-manifolds, J. Algebra 106 (2) (1987), 484-489.

[7] C. Droms, Isomorphisms of Graph Groups, Proc. Amer. Math. Soc. 100 (3) (1987), 407-408.

[8] C. Droms, Subgroups of graph groups, J. Algebra 110 (1987), 519-522.

[9] F. A. Garside, The braid group and other groups, Quart. J. Math. Oxford (2) 20 (1969), 235-54.

[10] W. Magnus, A. Karrass, and D. Solitar, Combinatorial Group Theory 2nd Revised Edition, Dover (1976).

[11] H. Servatius, Automorphisms of graph groups, J Algebra 126 (1) (1989), 34–60.

[12] B.A.F. Wehrfritz, Generalized free products of linear groups, Proc. London Math. Soc. (3) 27 (1973), 402-424.